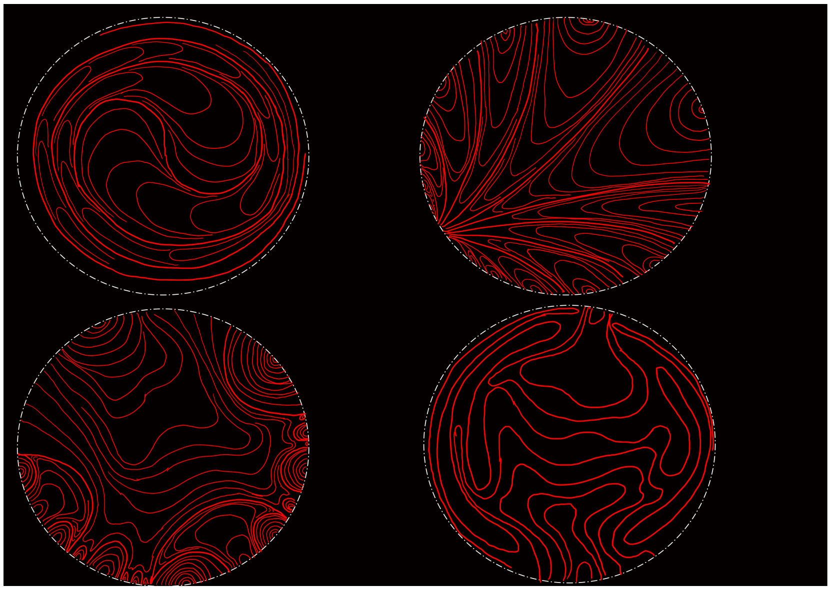
Foliations, commutative dynamics, and group actions on the line and the circle.



From my first foliation period (84-93) we can emphatize three results:

Theorem *For any compact surface S with non-zero Euler characteristic, there exists a neighborhood of the identity for the C^1-topology such that two commuting diffeomorphisms in this neighborhood always have a common fixed point.*

(see articles 14 and 17 below) This result (using only very basic techniques) resolved a conjecture (15 years old) by Harold Rosenberg. Because of its elementary side, it has been used in many Master's theses, all over the world. It was immediately generalized (in a non-elementary way) by Mr. Handel using Thurston's (pseudo-Anosov homeomorphisms) theory in a subttle way.

Theorem *Let F be a foliation defined by a fibration whose base is a compact manifold with non-zero Euler characteristic, and the fiber a compact manifold with a first real homology group equal to* ***R****. Then there exists a neighborhood of F for the C^1-topology of the foliation defined by this fibration, such that any foliation in this neighborhood has a compact sheet close to a fiber.*

(see articles 11, 15,16, 18, 19) This result ends to a series of works (Thurston, Rosenberg, Langevin, Schweitzer, Druck, Firmo), by resolving a Rosenberg conjecture. One of the essential tools comes from a theory of deformation, due to André Haefliger, and which we developed together (article 16).
In my opinion, here is my best result of this period:

Theorem (see 12) *Two commuting* **R**-*analytic vector fields of a 4-manifold with non-zero Euler characteristic always have a common zero.*

This result, very simply stated, seemed to open a way in a subject known to be difficult (see also the results of Molino and Turiel). It generalized a theorem of E. Lima on surfaces, and the same paper 12 presented a semi-local version in dimension 3. The reasons for stopping at dimension 4 seem purely technical.

The disinterest marked by the mathematical community for this last result, which seemed to me to be spectacular, was one of the causes of my thematic change towards a more "fashionable", and therefore also more active, subject, Dynamic Systems.

It is also necessary to cite article 10 with S. Firmo. It was, in a way, my farewell to foliations: it sets out a philosophy, is teeming with results, some ambitious, others less so, and cannot be summarized in a few lines. But a goodbye is not always final!!!
In conclusion, my results from this period in this field have opened some avenues which give hope for a quantitative local theory, such as the Poincaré-Hopf index, for compact sheets of foliations or the actions of finitely generated groups.

Back to foliations and group actions

Bruno Santiago came to Dijon to do part of his doctoral thesis with me, and he managed to convince me to take up the problem of the existence of common zeros for vector fields (C^3) in dimension 3. Sébastien Alvarez joined us… we have made good progress (articles 2 and 3 below) but the hoped-for result (see the conjecture below) still resists us.
Conjecture *Let* X,Y *be two vector fields of class* C^2 *of a manifold* M *of dimension 3, which commute. Let* U *be a relatively compact open end of* M *such that* Y *does not vanish on the boundary of* U. *The Poincaré-Hopf index of* Y *on* U *is therefore well defined and we assume that it is non-zero. Then* X *and* Y *have a common zero in* U.

Since then, the problem of group actions in dimension one, and therefore also foliations of codimension 1, has again become a very active problem, largely due to its links with the geometry of groups. See items 1, 4,5,6,7. Here is an example result:

Theorem (With Hélène Eynard-Bontemps, 5) *The space of actions of class* C^∞ *of* $\mathbb{Z}^n$ *on the interval is connected.*

List of my publications in foliations, commuting diffeomorphisms, and group actions in dimension 1

1. Bonatti, Christian; Carnevale, João; Triestino, Michele *Non-locally discrete actions on the circle with at most N fixed points*. **Math. Z.** 307, No. 1, Paper No. 6, 18 p. (2024).
2. Alvarez, Sébastien; Bonatti, Christian; Santiago, Bruno *Existence of common zeros for commuting vector fields on 3-manifolds. II. Solving global difficulties*. **Proc. Lond. Math. Soc.** (3) 121, No. 4, 828-875 (2020).
3. Bonatti, Christian; Santiago, Bruno *Existence of common zeros for commuting vector fields on three manifolds*. **Ann. Inst. Fourier** 67, No. 4, 1741-1781 (2017).
4. Bonatti, C.; Monteverde, Ignacio; Navas, Andres; Rivas, Cristobal *Rigidity for* C^1 *actions on the interval arising from hyperbolicity. I: Solvable groups*. **Math. Z.** 286, No. 3-4, 919-949 (2017).
5. Bonatti, Christian; Eynard-Bontemps, Hélène *Connectedness of the space of smooth actions of* $\mathbb{Z}^n$ *on the interval*. **Ergodic Theory Dyn. Syst**. 36, No. 7, 2076-2106 (2016).
6. Bonatti, Christian; Farinelli, Églantine *Centralizers of* C^1*-contractions of the half line.* **Groups Geom. Dyn.** 9, No. 3, 831-889 (2015).

7. Bonatti, Christian; Guelman, Nancy *Smooth conjugacy classes of circle diffeomorphisms with irrational rotation number*. **Fundam. Math.** 227, No. 2, 129-162 (2014).

8. Bonatti, Christian; Franks, John *A Hölder continuous vector field tangent to many foliations*. Brin, Michael (ed.) et al., **Modern dynamical systems and applications. Dedicated to Anatole Katok on his 60th birthday**. Cambridge: Cambridge University Press (ISBN 0-521-84073-2/hbk). 299-306 (2004).

9. Bonatti, Christian; Gómez-Mont, Xavier *Sur le comportement statistique des feuilles de certains feuilletages holomorphes.* Ghys, Étienne (ed.) et al., Essays on geometry and related topics. Mémoires dédiés à André Haefliger. Vol. 1. Genève: **L'Enseignement Mathématique. Monogr. Enseign. Math.** 38, 15-41 (2001).

10. Bonatti, Christian; Firmo, Sebastião *Feuilles compactes d'un feuilletage générique en codimension 1.*) **Ann. Sci. Éc. Norm. Supér.** (4) 27, No. 4, 407-461 (1994).

11. Bonatti, C. *Feuilletages proches d'une fibration.* **Ensaios Matemáticos 5**. Rio de Janeiro: Sociedade Brasileira de Matemática. v, 250 p. (1993).

12. Bonatti, Christian *Champs de vecteurs analytiques commutants, en dimension 3 ou 4: Existence de zeros communs*. **Bol. Soc. Bras. Mat.**, Nova Sér. 22, No. 2, 215-247 (1992).

13. Bonatti, C.; Langevin, R.; Moussu, R. *Feuilletages de CP(n) : De l'holonomie hyperbolique pour les minimaux exceptionnels.* **Publ. Math., Inst. Hautes Étud. Sci.** 75, 123-134 (1992).

14. Bonatti, Christian *Difféomorphismes commutants des surfaces et stabilité des fibrations en tores*. **Topology 29,** No. 1, 101-126 (1990).

15. Bonatti, Christian *Stabilité de feuilles compactes pour les feuilletages définis par des fibrations.* **Topology 29**, No. 2, 231-245 (1990).

16. Bonatti, C.; Haefliger, A. *Déformations de feuilletages*. **Topology 29**, No. 2, 205-229 (1990).

17. Bonatti, Christian *Un point fixe commun pour des difféomorphismes commutants de* S^2 **Ann. Math.** (2) 129, No. 1, 61-69 (1989).

18. Bonatti, Christian; Haefliger, Andre *Perturbations d'un feuilletage donné par une fibration: Existence de feuilles compactes.* **Proc. 15th Braz. Colloq. Math. Poços de Caldas**/Braz. 1985, 567 (1987).

19. Bonatti, Christian *Existence de feuilles compactes pour les feuilletages proches d'une fibration.* **C. R. Acad. Sci.,** Paris, Sér. I 305, 199-202 (1987).

20. Bonatti, Christian *Sur les feuilletages singuliers stables des variétés de dimension trois.* **Comment. Math. Helv.** 60, 429-444 (1985).

21. Bonatti, Christian *Existence de feuilletages singuliers de codimension un à feuilles denses sur les variétés compactes sans bord*. **C. R. Acad. Sci.,** Paris, Sér. I 300, 493-496 (1985).