# Weak hyperbolic structures and robust properties of diffeomorphisms and flows.

Christian Bonatti

CNRS & Université de Bourgogne

Berlin, 20th July 2016

Joint work with Adriana da Luz

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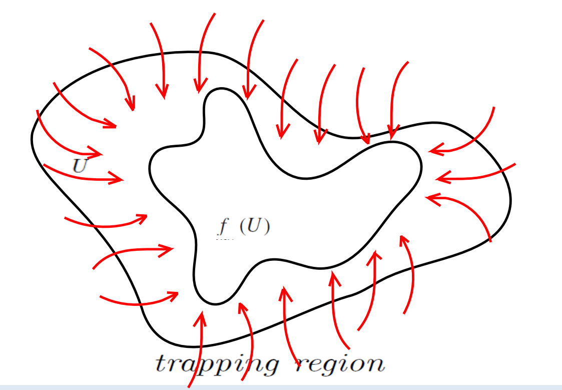
# Trapping region

*trapping region*: $U$ compact, $f(U)$ contained in $Int(U)$



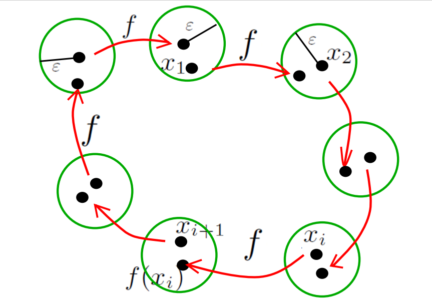
## Theorem (Conley theory (1978))

*$x$ chain recurrent $\Leftrightarrow$ for all $U$ trapping region,*

$$x \in U \Leftrightarrow f(x) \in U.$$

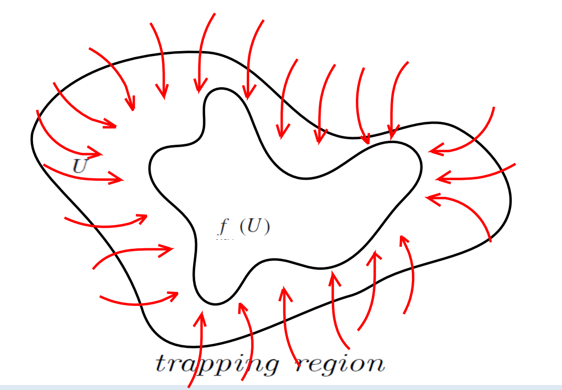
*$x, y$ in the same recurrence class $\Leftrightarrow$ for all $U$ trapping region,*

$$x \in U \Leftrightarrow y \in U$$



# Trapping region

*robust property*: $U$ is a trapping region for every $g$ close to $f$.



# Dominated splitting

$K$ invariant compact set,
$F \subset TM|_K$ a $Df$-invariant bundle.

- there are small trapping neighborhoods of $\mathbb{P}F$ in $\mathbb{P}M|_K$.

$$\Longleftrightarrow$$

- there is an invariant splitting $TM|_K = E \oplus F$, and $n > 0$ so that: $\forall$ unit vectors $u \in E_x$, $v \in F_x$ one has

$$\|Df^n(u)\| < \frac{1}{2}\|Df^n(v)\|.$$

$TM|_K = E \oplus_< F$ is a *dominated splitting*

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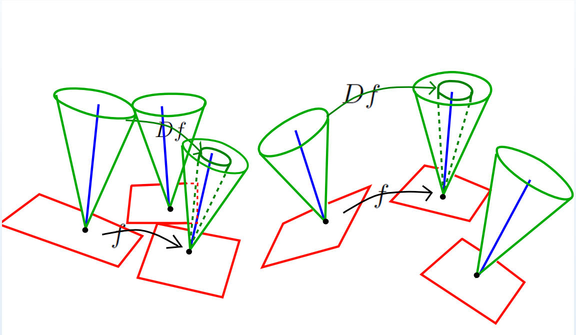


**Proposition (Robust property)**

*$\exists U$ neighborhoods of $K$, all $g$ $C^1$-close to $f$ admits*

$$E_g \oplus_< F_g$$

*on the maximal invariant set* $\Lambda(g, U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$.

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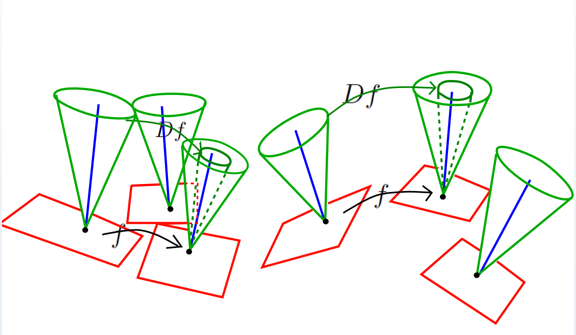


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# Finest dominated splitting

Dominated splitting with more than two bundles:
$\longrightarrow$ there is a unique *finest dominated splitting*.

$$E_1 \oplus_< E_2 \cdots \oplus_< E_k$$

dominated splitting=unique obstruction for mixing Lyapunov exponents. Mañé,Díaz,Pujals,Ures, B–,Wen,Gan...,Gourmelon,Vivier

**Theorem (Bochi,B–)**

*$K$ Hausdorff limit of $\gamma_n$ periodic orbits of period $\pi_n \to \infty$.*

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*Small perturbations of $Df$ over $\gamma_n \longrightarrow$*
*change the Lyapunov exponents in each $E_i$ by convex sums, keeping the sum in each $E_i$*

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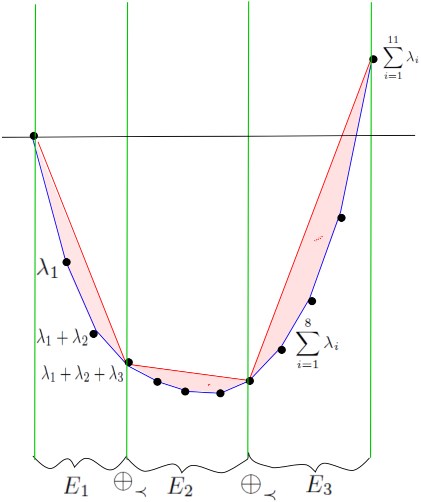
# Lyapunov exponents

Changing the Lyapunov Exponents in the bundles of the **finest dominated splitting**



# (Weak) hyperbolic structures

A *(weak) hyperbolic structure* on $K$ invariant is

- a dominated splitting $E_1 \oplus_< E_2 \cdots \oplus_< E_k$
- expansion/contraction of some quantity in some $E_i$.

examples:

- *hyperbolicity*: $E^s \oplus_< E^u$, (norm of vectors)
- *partial hyperbolicity*: $E^s \oplus_< E^c \oplus_< E^u$ or $E^s \oplus_< E^{cu}$, (norm of vectors)
- *volume partial hyperbolicity* $E^{cs} \oplus_< E^c \oplus_< E^{cu}$, (volume in $E^{cs}$ and $E^{cu}$)
- volume in subspaces of $dim = j$ in $E_i$ ($\longrightarrow$ at most $j$-1 negative exponents in $E_i$)

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# Weak hyperbolic structures/robust properties

A hyperbolic structure on $K$ invariant:

- is robust,
- restricts the effect of perturbations of the dynamics,
- $\longrightarrow$ helps for understanding robust properties

examples:

- hyperbolicity $\Leftrightarrow$ *$C^1$-structural stability*
  (Smale, Robbin, Robinson 1976, Mañé 1988, Palis)
- volume partial hyperbolicity $\Leftarrow$ *robust (chain) transitivity*.
  (Mañé 1982 ,[Díaz, Pujals, Ures 1999][B–, Díaz, Pujals 2003])

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# Vector fields

Vector fields in dimension $d \simeq$ Diffeomorphisms in dimension $d-1$?

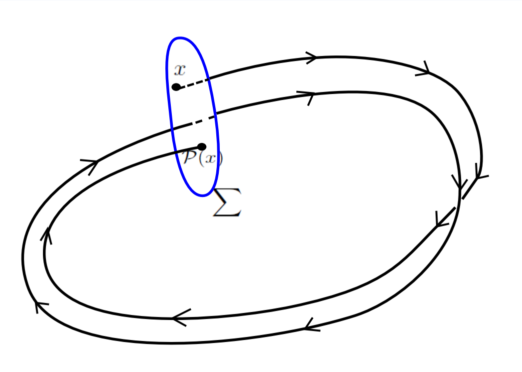


Figure : The Poincaré map on the cross section $\sum$

Vector field on surface: very similar to 1-dimensional dynamics

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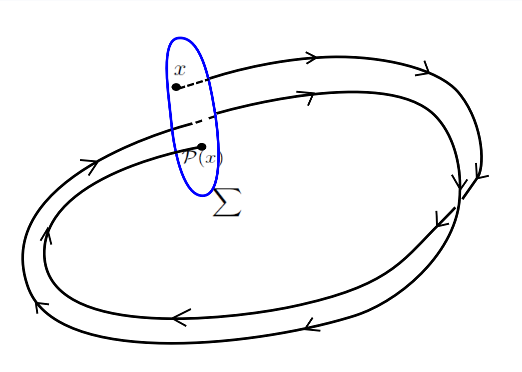
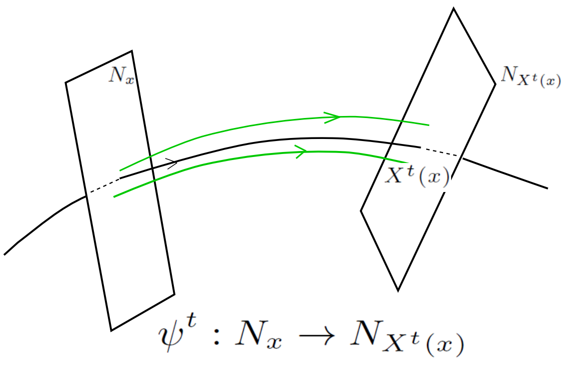



Figure : The Poincaré map on the cross section $\sum$

Vector field on surface: very similar to 1-dimensional dynamics.

# Far from singularities

$X$ vector field on $M$ compact. If $Zero(X)$ is isolated in $\mathcal{R}(X)$, hyperbolic structures are living on the *normal bundle*, for the *linear Poincaré flow* $\psi_t$.



# Hyperbolic structures/robust properties

Same results as for diffeomorphisms. For example:

**Theorem (Doering, Vivier)**

*$X$ robustly transitive $\Longrightarrow$ Zero$(X) = \emptyset$ and the linear Poincaré flow $\psi^t$ is volume partially hyperbolic.*

No dominated splitting for $X^t$!

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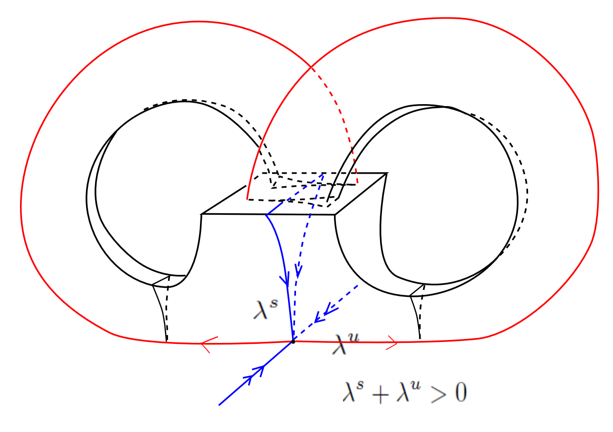
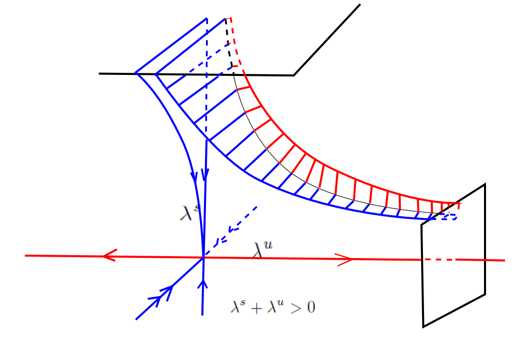
# Singular vector fields: Lorenz attractor

[Guckenheimer,Williams 1979] First example where
*Zero*($X$) is $C^1$-robustly accumulated by periodic orbits



# The linear Poincaré flow $\psi_t$ near the singular point

Dominated splitting $E^s \oplus_< F$.
$F$ is not uniformly expanded near the singularity,
but $C^1$-robust uniform expansion of the entrance/exit map



# Singular hyperbolicity

the flow $X^t$ admits a dominated splitting $E^s \oplus_< E^{cu}$:

- $E^s$ uniformly contracted
- the area is uniformly expanded on $E^{cu}$.

In terms of linear Poincaré flow $\psi^t$:
one multiplies $\psi^t|_F$ by the expansion $h^t$ of $X^t$ in the direction of $X$

$$h^t = \|DX^t|_{\mathbb{R}X}\|$$

The flow $(\psi^t|_{E^s}, h^t \cdot \psi^t|_F)$ is hyperbolic out of $\sigma$.

**Remark**

- *$h^t$ needs to satisy the cocycle condition*

$$h^{t+s}(x) = h^t(x)h^s(X^t(x)).$$

- *different kinds of singularity $\longrightarrow$ one needs local cocycles.*

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# cocycle associated to a singular point

**Theorem (B–,da Luz)**

*$\sigma \in Zero(X)$ hyperbolic. There is a cocycle $h\colon M \setminus \{\sigma\} \times \mathbb{R} \to ]0, +\infty[, (x,t) \mapsto h^t(x)$ continuous so that:*

- *if $x$ and $X^t(x)$ are far from $\sigma$ then $h^t(x) = 1$.*
- *there is $\|.\|$ so that, if $x$ and $X^t(x)$ are far from $Zero(X) \setminus \{\sigma\}$ then*

$$h^t(x) = \|DX^t|_{\mathbb{R}X(x)}\|$$

possible if $\|X\| = 1$ far from $Zero(X)$

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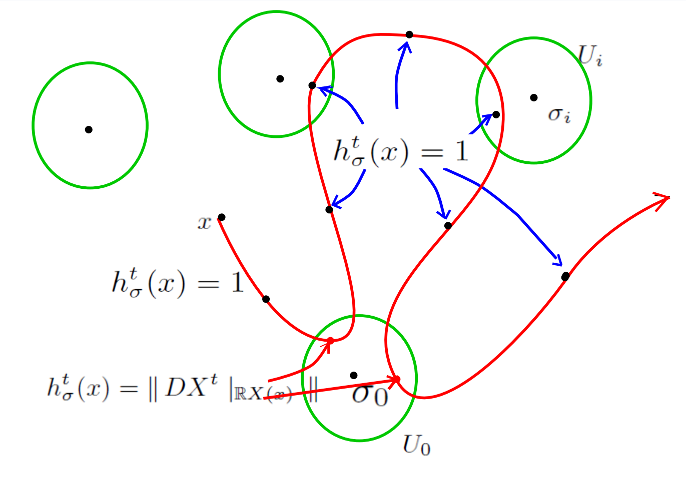
# Reparametrization cocycle associated to $\sigma$



# Reparametrization cocycles

**Theorem (continuation)**

- $h_{\sigma,X}$ is unique up to multiplication by a coboundary.
- $X \mapsto h_{\sigma,X}$ is continuous.

**Definition**

A *reparametrization cocycle* is a cocycle of the form

$$h^t = \prod_{\sigma \in Zero(X)} (h^t_\sigma)^{\alpha(\sigma)}$$

where $\alpha(\sigma) \in \mathbb{R}$

# Reparametrization cocycles

## Theorem (continuation)

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## Definition

*A reparametrization cocycle is a cocycle of the form*

$$h^t = \Pi_{\sigma \in Zero(X)} (h^t_\sigma)^{\alpha(\sigma)}$$

*where* $\alpha(\sigma) \in \mathbb{R}$

# Blowing up the singular points

$\mathbb{P}M \to M$ the bundle whose fiber over $x \in M$ is $\mathbb{P}T_xM$.

- The derivative $DX^t \longrightarrow$ topological flow $X^t_{\mathbb{P}}$ on $\mathbb{P}M$.
- $M \setminus Zero(X)$ is included in $\mathbb{P}M$ by $x \mapsto \mathbb{R}X(x)$.

Denote

$$\Delta_X = \{\mathbb{R}X(x), x \in M \setminus Zero(X)\} \cup \bigcup_{\sigma \in Zero(X)} \mathbb{P}T_\sigma M$$

- used by Liao, Wen, Gan,...
- also called *Nash blowing up* in algebraic geometry.
- the parametrizing cocycles extend on $\Delta_X$.

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# The extended linear Poincaré flow

The *extended linear Poincaré flow* on the normal bundle $\mathcal{N} \to \mathbb{P}M$.

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{\psi^t_{\mathcal{N}}} & \mathcal{N} \\ \downarrow & & \downarrow \\ \mathbb{P}M & \xrightarrow{X^t_{\mathbb{P}}} & \mathbb{P}M \\ \downarrow & & \downarrow \\ M & \xrightarrow{X^t} & M \end{array}$$

$\psi^t : N_L \to N_{X^t_{\mathbb{P}}(L)}$ is the quotient of $DX^t$ by $L$ and $X^t_{\mathbb{P}}(L)$.

# Singular hyperbolicities

## Definition

$K \subset \Delta_X$ is *singular volume partially hyperbolic* if

- $\psi^t_{\mathcal{N}}$ admits a dominated splitting $E^{cs} \oplus_< E^c \oplus_< E^{cu}$ over $\mathcal{N}|_K$.
- there are $h^t_{cs}$ and $h^t_{cu}$ so that
  - $h^t_{cs} \cdot \psi^t_{\mathcal{N}}$ contracts uniformly the volume in $E^{cs}$
  - $h^t_{cu} \cdot \psi^t_{\mathcal{N}}$ expands uniformly the volume in $E^{cu}$

*multisingular hyperbolic* (for avoiding confusion with the classical singular hyperbolicity)

*singular partially hyperbolic*

...

# On which set?

$U \subset M$ compact. $\Lambda(X, U)$ maximal invariant set in $U$.

$$\tilde{\Lambda}(X, U) = \limsup_{Y \to X} \overline{\{\mathbb{R}Y(x), x \in \Lambda(Y, U) \setminus Zero(Y)\}} \subset \Delta_X$$

**Definition**

*$\Lambda(X, U)$ has a singular weak hyperbolic structure if $\psi^t_{\mathcal{N}}$ has this structure over $\tilde{\Lambda}(X, U)$.*

- these notions are robust
- induce the usual notions far from singularities.

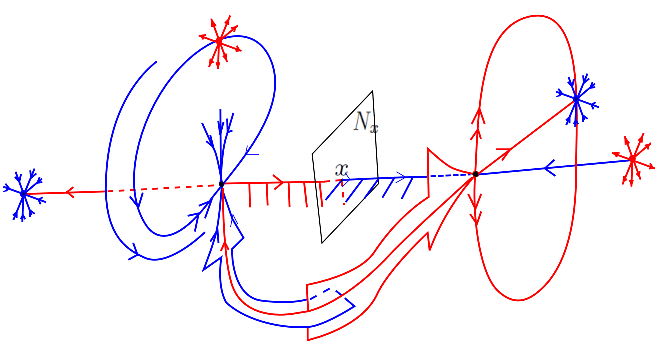
# Star flows

$X$ is *star* if every periodic orbit of every $Y$ $C^1$-close to $X$ is hyperbolic.

**Theorem ([Gan,Shi,Wen (2014)],[B–,da Luz (preprint)])**

*for an open and dense subset of star flows $X$, every chain recurrence class is multisingular hyperbolic.*

# A star flow which is not singular hyperbolic




This flow is multisingular hyperbolic, hence is star.

# An example

**Theorem (B–,da Luz)**

*On a 5-manifold, there is a $C^1$-open set of star flow with two singularities of different indices in the same chain recurrence class.*

These flow are multisingular hyperbolic but cannot be singular hyperbolic in the usual sense.

Thanks!

# A joint work

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