Robust tangencies

Preliminary version

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Abstract

Robust homoclinic tangencies is a local phenomenon generating non-hyperbolic behavior. It has been studied in extense for C^2-diffeomorphisms on surface, and for the C^2-topology. Nevertheless, C^1-robust tangencies exist in higher dimension; however, up to now, are associated with another robust phenomenon, the heterodimensional cycles.

- Recently, C.G. Moreira proved that C^1-robust tangencies does not exists in dimension 2: the proof of this deep result will be the main goal of this course.(lectures 4,5,6)
- Moreira's proof consist in separating two *dynamical Cantor sets* in the C^1-topology. As an introduction to Moreira's proof, I will recall the argument of Newhouse showing that thick dynamical Cantor sets cannot be separated by C^2-small perturbations (lectures 3 and 4)
- We will also present example of C^1-robust tangencies in higher dimensions, and a local mechanism providing C^1-robust tangencies, implying that this phenomenon exist in many non-hyperbolic systems (lectures 2 and 3)
- We start this course by trying to point out the importance of robust tangencies for giving a global picture, for the C^1-topology, of dynamical systems. (First lecture)

Contents

Warning

When I am sending this notes, they not yet finished or complete. It remains missprints and english mistakes, there are missing references, other which are not usefull here, etc.... However, it is a good indication of the content and the spirit of the mini-course.

I hope to send a more complete and clean version of my notes just afterthe conference.

1 Robust tangencies in a global view of C^1-dynamical systems

1.1 Approaching the global dynamics by periodic orbits

During the last 2 decades, there has been a lot of works explorating the dynamics of the diffeomorphisms or the vector-fields on compact manifolds, from the point of view of the C^1-topology:

- lemmas of C^1-perturbations of the orbits, as Pugh closing lemma and Hayashi connecting lemma, allowed us to show that the dynamics of C^1-generic diffeomorphisms (or flows) is very well approached by the periodic orbits:
  - the chain recurrent set is the closure of the set of periodic orbits ([BC]). More generally, every *chain transitive set*[1] is the Hausdorff limit of periodic orbits ([Cr]).
  - the chain recurrence classes containing a periodic orbit is the homoclinic class of the periodic orbit [BC];
  - every ergodic measure is the Hausdorff and weak limit of periodic measure [Ma]
- lemmas of C^1-perturbations of the local dynamics in the neighborhood of periodic orbits, through Franks lemma, relate the lack of hyperbolicity and dominated splittings with the bifurcations associated with periodic orbits.

[1] An invariant compact set K is *chain transitive* if one can goe from any $x \in K$ to any $y \in K$ by pseudo orbits in K with arbitrarily small jumps

1.2 Lack of hyperbolicity and periodic orbits

Consider dynamical systems far from hyperbolic dynamics: $f \in Diff^1(M) \setminus \overline{\{AxiomA + nocycle\}}$. As, C^1-generically, the global dynamic is very well aproached by periodic orbits, this lack of hyperbolicity is reflected by a lack of hyperbolicity on the periodic orbits (this are important ideas due to Mañé and Liao in the 70-80ies).

Let me try a first conjecture (first formulated in dimension 2 in [ABCD]): the robust non hyperbolicty is due to the robust non-hyperbolicity of a homoclinic class

Conjecture 1. *There is a dense open subset in $Diff^1(M) \setminus \overline{\{\text{Axiom A + no cycle}\}}$ of diffeomorphisms having a hyperbolic periodic point p_f whose homoclinic class (or chain recurrence class) is robustly non hyperbolic: the chain recurrence class $C(p_g)$ is not hyperbolic for every g in a C^1-neighborhood of f.*

This conjecture remains open in any dimension ≥ 2. In dimension 2, after Moreira's result, this conjecture remains the main difficulty for proving Smale's conjecture (the density of Axiom A diffeomorphisms on surfaces).

Remark 1.1. *If this conjecture is false, then there is an open set $\mathcal{U}$ of $Diff^1(M)$ such that for every C^1-generic diffeomorphism $f \in \mathcal{U}$ is* wilde hyperbolic *in the following sense:*

- *every homoclinic class is an hyperbolic basic set (in particular is isolated)*
- *there are infinitely many homoclinic classes, accumulating on aperiodic classes*

This would contradics the philosophy which is behind most of the approaches in non-hyperbolic dynamical systems: looking at the local phenomena (in general related to 1 *or finitely many periodic orbits) generating rich global behaviors.*

So we denote by $\mathcal{W}_{Hyp}(M)$ the maximal open set of $Diff^1(M)$ in which the generic diffeomorphisms are wild hyperbolic. Conjecture 1 states that

$$\mathcal{W}_{Hyp}(M) = \emptyset.$$

1.3 Lack of hyperbolicity and bifurcations

The two ways for loosing the uniform hyperbolicity on the set of periodic orbits are:

- either one looses the uniform exponential contraction/expansion at the period:

$$\lim_{n\to\infty} \frac{1}{per(x_n)} \log\left(\mathcal{M}\left(Df^{per(x_n)}|_{E^u(x_n)}\right) - \left\|Df^{per(x_n)}|_{E^s(x_n)}\right\|\right) = 0$$

- or one looses the uniform domination of the stable/unstable splitting along the orbits: they are arbitrarily large time intervals where the expansion in the unstable direction is not twice the expansion in the stable direction.

These two phenomena leads to two different kind of bifurcations:

- in the first case, up to a small perturbation, one direction changes from contracting to expanding or the contrary ([Ma_1]): in other words, one may perform a saddle node or a flip bifurcation. If this phenomena happens persistently in some open region of $Diff(M)$ then one has the *coexistence of different indices*, and one conjectures that this leads to *hetero-dimensional cycles*;
- in the second case, up to a small perturbation, the stable and unstable direction makes a very small angle: this may lead to *homoclinic tangency* ([PS, W_2, Go]).

This suggested the following conjecture, formulated by J Palis in any C^r-topology, but with many progresses in the C^1-topology:

Conjecture 2 (Palis density conjecture). *There is a dense open subset $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ of $Diff^1(M)$ such that $f \in \mathcal{O}_1$ statisfies the Axiom A without cycle, and there is a dense subset $\mathcal{D} \subset \mathcal{O}_2$ such that $f \in \mathcal{D}$ admits a heterodimensional cycle or a homoclinic tangency.*

For instance, the mini-course by Crovisier and Pujals will present their recent proof of the C^1-density conjecture for the attractors and repellers.

However, Kupka Smale theorem implies that for C^r generic diffeomorphisms the periodic orbits are hyperbolic, the stable and unstable manifolds are all transverse, so that f has no heterodimensional cycles nor homoclinic tangencies. So, Palis conjecture implies that, far from hyperbolic systems, even if perturbations can destroy all the cycles or the tangencies, new perturbations can rebuild (other?) cycles or (other?) tangencies.

In my mind, this means that the heterodimensional cycles and the homoclinic tangencies are not responsable for the robust non-hyperbolicity, but at the contrary, there are consequences of the lack of hyperbolicity.

For caracterizing the non-hyperbolicity, one would like to found *C^1-robust local phenomenon generating homoclinic tangencies and/or heterodimensional cycles.*

1.4 Robust cycles and tangencies

From Abraham-Smale examples, one knows the existence of C^1-robust cycle relating hyperbolic basic sets of different indices:

Definition 1.2. *Let $\mathcal{U}$ be a C^1-open set of diffeomorphisms f having hyperbolic basic sets K_f and L_f, varying continuously with $f \in \mathcal{U}$, such that the indices (dimension of the stable bundle) are differents , and such that $W^s(K_f) \cap W^u(L_f) \neq \emptyset$ and $W^u(K_f) \cap W^s(L_f) \neq \emptyset$.*

Then we say that f has a C^1-robust cycle *associated to K_f and L_f.*

If $f \in \mathcal{U}$ has a robust cycle associated to K_f and L_f and if $p_f \in K_f$ and $q_f \in L_f$ are hyperbolic periodic points (of different indices), then C^∞-densely in $\mathcal{U}$, f performs an heterodimensional cycle associated to p_f and q_f. Assume for instance that $dimE^s(q) < dim(E^s(p))$ so that $dim(E^s(p) + dimE^u(q) > dimM$. Then for an open and dense subset of $\mathcal{U}$, $W^s(p)$ cuts transversally $W^u(q)$ at some point. Now, small perturbations allow to get that $W^u(p)$ will cross quasi-transversally every stable manifolds in $W^s(L_f)$ and, densely, this stable manifold will be the one of q_f.

One defines robust tangencies in the same way :

Definition 1.3. *Let $\mathcal{U}$ be a C^1-open set of diffeomorphisms f having hyperbolic basic sets K_f varying continuously with $f \in \mathcal{U}$, such that $W^s(K_f) \cap W^u(K_f) \neq \emptyset$ contains a non-tranverse intersection point. Then we say that f has a C^1-robust tangency associated to K_f.*

Once again, robust tangencies associated to a hyperbolic basic set K_f , $f \in \mathcal{U}$, lead to a dense subset of $\mathcal{U}$ with homoclinic tangency associated to p_f, where p_f is any periodic point in K_f.

1.5 Existence of robust cycles and robust tangencies

In 68, [AS] built the first example of a C^1-open set of non-Axom A diffeomorphisms, on a 4-manifold. Then in 72,[Si] built an example in dimension 3. These examples consisted in building a robust heterodimensional cycle. As recently pointed out by Asaoka [As], their construction leads also to a robust tangency. I will explain a similar construction in the next chapter.

In 74 [N_3] built a C^2-open set of diffeomorphisms on surfaces having a C^2-robust tangency associated to a hyperbolic basic set Λ, assuming that Λ is *thick*: the product of its stable thickeness and of its unstable thickness is larger than 1. Furthermore, he proved that every homoclinic tangency associated to a periodic point p generates, by performing the bifurcation, a thick hyperbolic set related to p and having a homoclinic tangency: so every tangency can be turn robust (see [N_3]) !

However, Newhouse result holds in dimension 2, and that just for C^r-topology, $r > 1$. There are generalisation in special cases in higher dimension (see [PV]), for the C^r-topology, $r > 1$.

1.6 From cycle and homoclinic tangency to robust cycles or tangencies

Palis conjecture would give an explanation of the non-hyperbolic dynamics if it was possible to turn robust every heterodimensional cycle and homoclinic tangency.

Indeed it is almost done for heterodimensional cycles:

Theorem 1.1. *[BD_4] If f is a diffeomorphism admiting a heterodimensional cycle associated to periodic points p,q wit $ind(p) - ind(q) = 1$ then there is g close to f having a robust cycle.*

(in most of the cases, one may ensure that the robust cycle is associated to the continuation of p and q but there is precisely one configuration where we could build counterexample).

Is it possible to turn robust a homoclinic tangency? We will see that Moreira's result answer negatively to this question: in dimension 2 there are no C^1-robust tangency.

Notice that robust (or persistent) tangencies associated to a periodic point p leads to accumulations of periodic orbits of a different index in a neighborhood of the homoclinic class of p: every homoclinic tangency associated to p generates perioodic orbits having a complex eigenvalues corresponding the the weakest stable and unstable eigenvalues of p. Hence it is natural to espect that robust tangency leads to heterodimensional cycles and to robust cycles.

Conjecture 3 (Bonatti)**.** *Let $\mathcal{U}$ being a C^1-open set of diffeomorphisms f having a hyperbolic basic set K_f varying continuousliy with f and presenting a robust tangency. Then there is a C^1-dense open subset $\mathcal{U}_1$ of $\mathcal{U}$ such that for f in $\mathcal{U}_1$ there is a hyperbolic basic set L_f of different index as K_f and such that (K_f, L_f) present a robust cycle.*

In dimension 2, there are no robust cycles, so this conjecture means that there are no robust tangency, which is the recent result by Moreira (see section 2.2). In higher dimension (with Crovisier, Diaz and Gourmelon) we have very partial results in this direction (see section 2.3).

This conjecture would be an important step in another conjecture, which generalizes Palis density conjecture:

Conjecture 4 ([BD_4])**.** *The union of the disjoint C^1-open sets of diffemorphisms $\mathcal{H} \cup \mathcal{RC}$, where $\mathcal{H}$ is the set of Axiom A + no cycle diffeomorphisms and $\mathcal{RC}$ is the set of diffeomorphism presenting a robust cycle, is dense in $Diff^1(M)$.*

Remark 1.4.

- *This conjecture provides a caracterization of the non hyperbolicity which would be checkable by computers: being Axiom A + no-cycle is algoritmically checkable and having a robust cycle is checkable too.*
- *This conjecture point out the heterodimensional cycles has the unique culprit of the robust non-hyperbolicity. Does it mean that the robust tangency has no role in that theory? Next conjecture point out the robust tangencies as necessary for the wild behaviors.*

1.7 Tame and wild dynamics

1.7.1 Definitions

Using an argument of genericity (using Pugh closing lemma), the fact that periodic orbits can be turned hyperbolic, and Conley theory, one can show show (see for instance [Ab, BC]):

Theorem 1.2. *There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that for $f \in \mathcal{R}$, every isolated chain recurrence class is robustly isolated (and is a homoclinic class).*

This leads to the natural notion:

Definition 1.5. *A diffeomorphism f is* tame *if every chain recurrence class is robustly isolated.*

One denotes by $\mathcal{T}(M)$ the set of tames diffeomorphism. It is a C^1 open set containing Axiom A + no cycle. A tame diffeomorphism has finitely many chain recurrence classes, and this number is locally constant.

A diffeomorphism is *wild* if it is far from tame diffeomorphisms. One denotes

$$\mathcal{W}(M) = Diff^1(M) \setminus \overline{\mathcal{T}(M)}$$

the set of *wild diffeomorphism.*

C^1-generic wild diffeomorphisms have infinitely many chain recurrence classes and infinitely many homoclinic classes.

1.7.2 Wild dynamics and wild homoclinic classes

My feeling is that wild dynamics are produced by a homoclinic class which generates new homoclinic classes nearby by perturbations. That is, once again, the wild behavior is seen from the periodic orbits, or better said, *the wild behavior is generated by a robust local phenomenon related to periodic orbits.* This may be expressed by the following conjecture:

Conjecture 5 (Bonatti)**.** *There is a dense open subset $\mathcal{O}$ of $\mathcal{W}(M)$ of diffeomorphism f having a hyperbolic periodic point p_f varying continuously with f, and such that for C^1-generic $f \in \mathcal{O}$ the homoclinic class $H(p_f, f)$ is not isolated.*

This leads to the notion of *wild homoclinic class*: One says that the homoclinic class $H(p_f, f)$ is a *wild homoclinic class* if it is *robustly non isolated*, that is, if for C^1-generic g close to f the class $H(p_g, g)$ is not isolated.

Using the fact that, for C^1-generic diffeomorphisms, isolated classes are robustly isolated and the fact that the number of homoclinic classes is countable, one proves easily

Lemme 1.6. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that if $f \in \mathcal{R}$ then every homoclinic class $H(p_f, f)$ which is not isolated is a wild homoclinic class.*

So Conjecture 5 may be restated as

Conjecture 6. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that every $f \in \mathcal{R} \cap \mathcal{W}(M)$ has a wild homoclinic class.*

Remark 1.7. *If this conjecture is wrong, then there is a non-empty open subset $\mathcal{U} \subset Diff^1(M)$ such that, for every C^1-generic diffeomorphisms $f \in \mathcal{U}$ one has:*

- *every homoclinic class is robuslty isolated*
- *there are sequence of homoclinic classes accumulating on aperiodic classes.*

1.7.3 wild homoclinic classes and robust tangencies

Here is the role of robust tangency:

Conjecture 7 (Bonatti)**.** *If $\mathcal{U}$ is an open set where p_f is a periodic point varying continuously with $f \in \mathcal{U}$ and $H(p_f, f)$ is a wild homoclinic class, then there is a dense open subset of $\mathcal{U}$ where $H(p_f, f)$ contains a robust tangency.*

The easier step for proving this conjecture is the next conjecture (first expressed at UMALCA Cancun (2004))

Conjecture 8 (Bonatti)**.**
1. *(weak version) There is a residual subset $\mathcal{R}$ fo $Diff^1(M)$ such that, for every $f \in \mathcal{R}$, every chain recurrence class admiting a partially hyperbolic splitting*

$$E^{ss} \oplus_{<} E^{c} \oplus_{<} E^{uu},$$

where $dim E^c = 1$, is isolated.

2. *(strong version) There is a residual subset $\mathcal{R}$ fo $Diff^1(M)$ such that, for every $f \in \mathcal{R}$, every chain recurrence class admiting a dominated splitting*

$$E^{ss} \oplus_< E_1^c \oplus_< \cdots \oplus_< E_k^c \oplus_< E^{uu},$$

where $dimE_i^c = 1$, is isolated.

This conjecture expresses that, if a dominated splitting forbids homoclinic tangencies, then the local dynamic is tame.

However, Conjecture 7 is far to provide a caracterization of wild dynamics, as there are example of tame dymanics which presents robust homoclinic tangencies.

Problem 1. *Find a characterization of tame diffeomorphisms, and of wild diffeomorphisms.*

It would be interesting to state a conjecture, characterizing tame and wild diffeomorphisms by using either a local phenomenon or a global structure. At this time, I am not able to propose such a conjecture.

1.8 Splitting $Diff^1(M)$ in 8 open regions

We consider 3 criteria:

- being robustly aproximated by heterodimensional cycles , or being far from heterodimensional cycles. This defines two disjoint open sets, whose union is dense.
- being robustly aproximated by homoclinic tangency, or being far from homoclinic tangency. This defines two disjoint open sets, whose union is dense.
- being wild or tame.

These criteria define 8 disjoint open regions whose union is dense in $Diff^1(M)$.

1. Tame diffeomorphisms far from homoclinic tangency and heterodimensional cycle are Axiom A + no cycle.
2. $$\{\text{Tame diffeomorphisms with tangency but far from cycles}\} = \emptyset$$
3. there are examples of tame diffeomorphisms far from tangency but with robust cycles.
4. there are example of tame diffeomorphisms with robust cycle and robust tangency
5. Palis density conjecture means that
$$\{\text{wild diffeomorphisms far from tangencies and cycles}\} = \emptyset.$$
6. Conjecture 4 is already known on tame diffeomorphisms. The open part of this conjecture means that
$$\{\text{wild diffeomorphisms far from cycles}\} = \emptyset.$$
7. Conjectures 5 and 7 mean that
$$\{\text{wild diffeomorphisms far from tangencies}\} = \emptyset.$$
8. there are examples of wild diffeomorphisms, using wild homoclinic classes having robust cycles and robust tangencies.

Summarizing, if all these conjectures are verified, one splits $Diff^1(M)$ in 4 open regions whose union is dense:

1. The AxiomA + no Cycles, which admits now a very complete description

2. The non-hyperbolic tame diffeomorphisms far from tangencies; every class admits a partially hyperbolic dominated splitting

$$E^{ss} \oplus_< E^c_1 \oplus_< \cdots \oplus_< E^c_k \oplus_< E^{uu}$$

where E^{ss} are non trivial and hyperbolic and $dimE^c_i = 1$. This class of diffeomorphisms seems to be ready for deeper studies, like exoistence and finitenes of SRB measures or of measure which maximalise the entropie....From the topological point of view, it remains two essential questions:

Problem 2. *Is it true that there is a dense open subset of the set of tame diffeomorphisms far from tangency, such that every chain recurrence class is robustly transitive? and is robustly a homoclinic class?*

The second consist in proving Conjecture 8:

Problem 3. *Let $\mathcal{U}$ be an open set where every chain recurrence class admits a partially hyperbolic dominated splitting*

$$E^{ss} \oplus_< E^c_1 \oplus_< \cdots \oplus_< E^c_k \oplus_< E^{uu}$$

where E^{ss} are non trivial and hyperbolic and $dimE^c_i = 1$.

Is $\mathcal{T}(M) \cap_c U$ dense in $\mathcal{U}$?

A positive answer to this question would provides a characterization of this class of diffeomorphisms.

3. The non-hyperbolic tame diffeomorphisms with robust tangencies. This class is less understood. A deeper understanding of examples, like Carvalho's attractor or Bonatti-Viana robusly transitive diffeomorphisms could be very helpfull. In some case, the ergodic study has been successfully performed

Add references: (Carvalho, Tahzibi-Hertz...)

4. The wild diffeomorphisms. It is certainly the less understood region. We just know some examples. See next section where I propose conjectures on wild dynamics.

1.9 Wild dynamics

Section far from the main subject: not presented in the course

1.9.1 Universal and k-universal dynamics

In [BD$_3$] we build a robust local mechanisms, which is a homoclinic class, contained in a ball and isotopic to the identity, which generates, by perturbation, periodic orbits whose derivative is the identity. Hence small perturbation generates small periodic discs on which the return map is the identity. Hence a new perturbation produce our local mechanisms inside these discs, and so on. These has two consequences:

- as the identity map generates by perturbation any orientation preserving diffeomorphisms on the disk, generic diffeomorphisms in that open set have a *universal dynamics*: renormalizing the dynamics in the (disjoint) periodic discs, one gets a dense subset of $Diff^1_0({}_D D^n)$.
- our phenomenon is a wild homoclinic class $H(p)$ which generates by perturbation the same kind of wild homoclinic classes outside $H(p)$: this leads to a non-countable set of classes. As the homoclinic classes are countably many, these local generic diffeomorphisms have *aperiodic classes*

One can generalize this process, in dimensions lower than $dimM$. Iff a homoclinic class produces by perturbation isolated periodic orbits having $k \geq 3$ eigenvalues arbitrarily close to 1, then a small pretubation will produce a filtrating set on which the maximal invairant set will be contained in normally hyperrbolic k-disk, in restriction to which the dynamics is universal. Once more, this leads to, locally generically, uncountably many aperiodic classes.

Let $\widetilde{Diff}^1(\mathbb{D}^k)$ denote the set of diffeomorphisms $\varphi\colon \mathbb{D}^k \to \mathbb{R}^k$ such that the maximal invariant set in $\mathbb{D}^k$ admits a fitrating neighborhood (intersection of an attracting region with repelling region). We denote by $\widetilde{Diff}^1_0(\mathbb{D}^k)$ the subset of those which are isotopic to the identity map.

Definition 1.8. *1. We say that a diffeomorphisms f has the k-universal dynamics if for every open set $\mathcal{O} \in \widetilde{Diff}^1_0(\mathbb{D}^k)$, there is $\pi \in \mathbb{N}$ and a subdisc $D \subset M$ of dimension k and period π such that*

- *the restriction $f^\pi|_D$ is smoothly conjugated to a diffeomorphism $\varphi \in \mathcal{O}$*
- *D is normally hyperbolic*

2. We say that f has the free k-universal dynamics *or* is freely k-universal *if furthermore there is a filtrating neighborhood $U \subset M$ of D such that the maximal invariant set of f in U is contained in the orbit of the maximal invariant set of f^π in D.*

Proposition 1.9. *1. For every $k \in \mathbb{N}^*$, the (free or not) $k+1$-universal dynamics implies the (free or not) k-universal dynamics.*

2. The (free or not) k-universal dynamics is a G_δ-property

3. if a C^1-generic diffeomorphism f has the free k-universal property, with $k \geq 3$, then it admits a pair (U, D) (where D is the normally hyperbolic k-disc and U the filtrating set U in the definition) such that the restriction $f^\pi|_D$ is k-universal.

4. if a C^1-generic diffeomorphism f has the free k-universal property, with $k \geq 3$, then it has uncountably many chain recurrence classes and aperiodic classes (contained in the orbit of a normally hyperbolic k-disc D in a filtrating set).

Proof : Using the fact that $Diff^1(\mathbb{D}^k)$ admits a countable basis of the topology, that the disk in the definition are normally hyperbolic hence persist by perturbation, that filtrating sets persist by perturbation, one get the G_δ property.

The item 3) implies the item 4), as in [BD_3]. □

We don't know if there are C^1-generic diffeomorphisms φ on surfaces with the free 2-universal dynamics (this is related to Smale conjecture, and Moreira's result). For this reason item 3 and 4 requires that $k \geq 3$.

1.9.2 Newhouse phenomenon

Newhouse built C^2-open sets of diffeomorphisms where a homoclinic class presents a robust tangency. The enfolding of this tangencies generates attracting dics, on which the diffeomorphisms presents a robust tangency. As in the case of universal dynamics, this leads to uncountably many aperiodic classes (see [BDV, Chapter???]. However, if the Jacobian was uniformly less than one, these C^2-robust tangencies cannot lead to universal dynamics

1.9.3 self reproducing properties

This two examples above leads to the (not yet well defined) notion of *self-reproducing* property . It consist in a filtrating set U whose (non-empty) maximal invariant set presents a robust local phenomenon P and such that the phenomenon P generates, by arbitrary small perturbation, two disjoint filtrating sets $U_1, U_2 \subset U$ whose maximal invariant sets presents the robust phenomena P.

contagious property?

Example 1. *For any $k \geq 3$, the k-universal dynamics on a normally hyperbolic k-submanifold is self reproducing.*

If a C^1-generic diffeomorphism has a self reproducing property P, then it has uncountably many chain recurrent classes.

Conjecture 9 (Bonatti)**.** *If a C^1-generic diffeomorphism has an aperiodic class, then it has uncountably many classes.*

Conjecture 10 (Bonatti)**.** *If a C^1-generic diffeomorphisms has uncountably many classes, it has a self-reproducing property stated on homoclinic classes (I will say shortly a self reproducing homoclinic class).*

As I did not defined correctly the self reproducing property, let me propose a more concrete problem, which illustrates the notion of self reproducing property:

Conjecture 11 (Bonatti)**.** *Let f be a diffeomorphism having a chain recurrence class $C(p_f)$, where p_f is a hyperbolic periodic point of f, such that:*

- *the class $C(f)$ contains two hyperbolic basic sets K_f and L_f related by a robust heterodimensional cycle and $p_f \in K_f$;*
- *the class $C(f)$ is* robustly without dominated splitting*: for every g in a neighborhood of f, $C(p_g)$ has no dominated splitting.*
- *the jacobian $|\det Df^{\pi(p)}(p)|$, where π is the period of p, is less than 1.*

Then, given any neighborhood U of $C(p_f)$ there is g arbitrarily C^1-close to f such that g admits an attracting region $W \subset U$ with $W \cap C(p_g) = \emptyset$ and W contains a chain recurrence class $C(q_g)$ which is robustly without dominated splitting.

Roughly speaking this conjecture asserts that the robust lack of dominated splitting is self reproducing.

1.9.4 Countable number of classes

At this time, there are no examples of C^1-open set of diffeomorphisms in which C^1-generic diffeomorphisms have countably many chain recurrence classes.

Let us say that a homoclinic class $H(p)$ is a *class with dandruff* if it admits a filtrating neighborhood U in which C^1-generic diffeomorphism g close to f have infinitely many classes in U, all the classes are isolated but the continuation $H(P_g)$. In other words, $H(P_g)$ generates classes outside $H(P_g)$ but only isolated classes.

Find a more politically correct word for this property

At this time, there are no examples which are known to be classes with dandruff. However,

Conjecture 12. *If $f \in \mathcal{W}(M)$ is a C^1-generic diffeomorphisms having countably many chain recurrence classes, then it has a homoclinic class with dandruff.*

1.10 Spliting the set of wild dynamical systems in regions

I would like to splitt $\mathcal{W}(M)$ in disjoint open subsets $\mathcal{W}_{Hyp}(M) \cup \mathcal{W}_{\mathbb{N}}(M)\mathcal{W}_{\mathbb{R}}(M)$ whose union is dense in $\mathcal{W}(M)$ such that

for every C^1-generic diffeomorphism f,

1. if $f \in \mathcal{W}_{hyp}(M)$ then f is wild hyperbolic; maybe be we need to devide this case in countable and uncountable set of chain recurrence classes.
2. if $f \in \mathcal{W}_{\mathbb{R}}$ then f has a self reproducing homoclinic class; in that case the set of classes of f is uncountable;
3. if $f \in \mathcal{W}_{\mathbb{N}}$ then f has a homoclinic class with dandruff, and the number of classes is countable;

Some last comments:

- Conjecture 1 states that $\mathcal{W}_{hyp}$ is empty and I don't know is $\mathcal{W}_{\mathbb{N}}$ is empty or not.

- For $\mathcal{W}_{\mathbb{R}}$ I would splitt it in open sets with the free k-universal dynamics for $k \geq 3$.
  - If Smale conjecture on surface is wrong, we would include $k \geq 2$.
  - for free $dimM$-universal dynamics, I would consider the study finished: f display infinitely times any generic phenomenon.
  - for free k-universal dynamics, $k < dimM$ the main study would concern attractors.
- With D. Yang, we think we have an example of a C^1-generic diffeomorphisms with a non-isolated partially hyperbolic class (in fact a quasi attractor) in dimension 3 with a splitting $E^{cs} \oplus E^{uu}$ with $dim(E^{uu} = 1$ and the jacobian in E^{cs} is < 1.

  This example cannot be universal for any $k > 1$. We don't know if it has countably or uncountably many classes.

1.10.1 Density of finitely many classes?

As noticed before, wild diffeomorphisms are not defined by themself: we just know that C^1-generic wild diffeomorphisms have infinitely many classes. This leads to a natural question

Question 1. *Does it exist a non-empty open subset $\mathcal{O} \subset \mathcal{W}(M)$ where every $f \in \mathcal{O}$ has infinitely many chain recurrence classes?*

A negative answer to this question would be a good indication for a Conjecture by Jacob Palis

Conjecture 13. *There is a dense subset of $Diff^r(M)$ of diffeomorphisms having finitely many attractors. Furthermore the union of the basins is dense in M.*

In case of a positive answer, we could stronger the question:

Question 2. *Does it exist a non-empty open subset $\mathcal{O} \subset \mathcal{W}(M)$ where every $f \in \mathcal{O}$ has infinitely many quasi-attractors?*

1.10.2 Attractors

In [BLY] we proved the existence of non-empty open sets where C^1-generric diffeomorphisms have no attractors nor repellers. According to Hurley, a quasi-attractor is a chain recurrence class admitting a basis of neighborhoods by attracting regions. [BC] proved that the ω-limit set of generic points of C^1-generic diffeomorphisms are quasi attractors.

Let call *generic attractor* every quasi-attractor whose basin contains a residual subset of a neighborhood (if the conjecture below is wrong, one could weaken the definition, just asking that the basin contains a residual subset in a non trivial open set).

Conjecture 14. *There is a residual subset $\mathcal{R}$ of $Diff^1(M)$ such that for every $f \in \mathcal{R}$ one has:*

1. *every generic attractor of f is a homoclinic class $H(p)$;*
2. *the union of the basins of the generic attractors is residual in M;*
3. *if $H(p_f)$ is a generic attractor for f, then $H(p_g)$ is a generic attractor for every C^1-generic g close to f;*
4. *the basin varies semi-continuously: given an attracting region V in the closure of the basin of f, then V it is contained in the closure of the basin of $H(p_g, g)$ for C^1-generic g close to f.*

2 Recent results on robust tangencies

2.1 abundance of robust tangencies

[BD_4] shows that one can turn robust any homoclinic tangency which occurs on a period point of a robust heterodimensional cycle. Let us state here some consequences:

Theorem 2.1. *There is a residual subset $\mathcal{R} \subset Diff^1(M)$ such that if $f \in \mathcal{R}$ and p is a periodic point of f such that $H(p)$ contains a periodic point of different index as p and the stable/unstable splitting over the periodic point homoclinically related with p is not dominated, then p belongs to a hyperbolic basic set having a robust tangency.*

As always, such perturbation lemma has strongest consequences on tame diffeomorphisms:

Corollaire 2.1. *Let $\mathcal{T}(M)$ be the C^1 open set of tame diffeomorphisms. Then there is an open an dense subset $\mathcal{O} \subset \mathcal{T}(M)$ such that, for every $f \in \mathcal{O}$ and every chain recurrecne class C of f one has*

- *either there is a partially hyperbolic splitting on C*

$$TM|_C = E^s \oplus_< E_1 \oplus_< \cdots \oplus_< E_k \oplus_< E^u$$

  where E^s is uniformly contracting, E^u is uniformly expanding and $dim(E_i) = 1$.

- *or C contains a hyperbolic basic set having a robust tangency; furthermore C contains a robust heterodimensional cycle.*

2.2 No robust tangency in dimension 2

Let us now state Moreira's result.

Theorem 2.2. *Let S be a closed surface. There are no C^1-robust tangencies for diffeomorphisms in $Diff^1(S)$.*

More precisely: given an open subset $\mathcal{O}$ of $Diff^1(M)$ where there is a hyperbolic basic set K_f varying continuously with f. Let $W^s_\ell(K_f)$ and $W^u_\ell(K_f)$ denote the local stable and unstable manifolds of length ℓ of K_f, that is, the union of the segments of length 2ℓ of stable or unstable manifolds centered at the points of K_f. Then

Theorem 2.3. *There is an open and dense subset of $\mathcal{O}$ where $W^s_\ell(K_f)$ is transverse to $W^u_\ell(K_f)$.*

This is an important step in direction of Smale's conjecture

Conjecture 15 (Smale)**.** *The Axiom A + no cycle diffeomorphisms are dense in $Diff^1(S)$.*

According to [ABCD] it remains 2 difficulties for proving Smale conjecture.

- one is given by Conjecture 1 stated above: diffeomorphisms having a C^1-robustly non-hyperbolic homoclinic classe are dense far from Axiom A + no cycle
- the second is that the non-hyperbolicity of a homoclinic class could not be seen, generically, on the intersection of the invariant manifolds of a hyperbolic set contained in the homoclinic class:

  Conjecture 16 ([ABCD])**.** *For f C^1-generic, if $H(p)$ is a non-hyperbolic homoclinic class then it contains a robust cycle or a robust tangency.*

2.3 Homoclinic tangencies and heterodimensional cycles in higher dimension

The results in this section are a work in progress with S. Crovisier, L. Díaz and N. Gourmelon.

Theorem 2.4. *Given P a hyperbolic periodic saddle with index $i \geq 2$. Assume that there is no nominated splitting on $H(P)$ neither of index $i-1$ nor of index i. Assume also that there is $Q \sim P$ such that $|\lambda_i(Q)\lambda_{i+1}(Q)| \geq 1$.*

Then there are arbitrarily small perturbations of f creating heterodimensional cycle between P_g and a point R_g of index $i-1$.

The hypothesis of *no dominated splitting of index i* is equivalent to *one can create a homoclinic tangency associated to p, by small C^1-perturbation* according to [W$_2$, Go].

Corollaire 2.2. *Given P a hyperbolic periodic saddle with index $2 \leq i \leq dimM-2$. Assume that there is no nominated splitting on $H(P)$ neither of index $i-1$ nor of index i nor of index $i+1$.*

Then there are arbitrarily small perturbations of f creating heterodimensional cycle between P_g and a point R_g of index $i-1$ or $i+1$.

2.4 Generalizing Moreira's result in higher dimension

Let me end this introduction by proposing a problem to the reader. It consists to try to generalize Moreira's result in higher dimension.

Conjecture 17 (Bonatti)**.** *Let K be an index $dim(M)-1$ hyperbolic basic set of a diffeomorphism $f\colon M \to M$. Assume furthermore that K is sectionally dissipative: for every $x \in K$, and every 2-plane $P \subset T_xM$ the determinant of the restriction of D_xf to P is less than 1:*

$$|\det(D_xf)|_P| < 1.$$

Then there is no robust tangency associated to K.

The proof of that conjecture could consists, as in Moreira'theorem, to separate a dynamical Cantor set in $\mathbb{R}^{dimM-1}$ from the product by $\mathbb{R}^{dimM-2}$ of a 1-dimensional dynamical Cantor set. However, we will see (see Theorem 3.2) that there are codimension 1 hyperbolic basic sets with robust tangency. That is, the hypothesis *sectionally dissipative* is essential in this conjecture. This hypothesis is not easy to include in the problem above of separating the two sets.

It is easy to see that this conjecture is wrong if K is not a Cantor set, and Theorem 3.2 also implies that it is wrong if K is a blender horseshoe. So when I was writing this notes, I completed this conjecture asking

Problem 4. *Let K be a sectionnally dissipative index $dim(M)-1$ hyperbolic basic set of a diffeomorphism $f\colon M \to M$. Prove that K is a Cantor set and is not a blender.*

However, today, Sylvain Crovisier gave a simple argument which seems to a positive answer to this problem, proving that the Hausdorff dimension of a sectionally dissipative Cantor set is less than one. That is a good starting point for this generalization Gugu's result.

3 C^1-robust tangencies in dimension larger than or equal to 3

3.1 C^1-robust heteroclinic tangency

Robust-heteroclinic tangencies are known from the sixties: they are responsable of the terminology *strong transversality condition* which is necessary for the structural stability. The idea for building robust heterodimensional tangencies is very simple:

Remark 3.1. *Thom's transversality theorem asserts that, generically, two submanifolds are always transversal. However, if you consider a foliation $\mathcal{F}$ and a submanifold N, one cannot apply Thom's theorem for putting N tranverse to $\mathcal{F}$. The easier reason is that, if N has the dimension of the codimension of $\mathcal{F}$, then if N cuts the leaves of $\mathcal{F}$ in two points with oposite orientations; then this property persists by C^1 perturbation of N and $\mathcal{F}$, and in some sense, by C^0 perturbation of N and $\mathcal{F}$: C^0 perturbation of N and $\mathcal{F}$ cannot put N transverse to $\mathcal{F}$.*

We will apply this simple remark to the intersection of a unstable manifold of a saddle with the stable foliation defined on the basin of an attractor. More precisely:

Consider a diffeomorphism $f \in Diff^1(M)$ having a non-trivial hyperbolic attractor $\mathcal{A}$; that is, $\mathcal{A}$ is a hyperbolic basic set, non reduced to a hyperbolic periodic sink, whose stable manifold contains a neighborhood of $\mathcal{A}$. The index of $\mathcal{A}$ is the dimension of the stable bundle of $\mathcal{A}$, hence is the stable index of the periodic saddle points contained in $\mathcal{A}$.

Recall that $W^s(\mathcal{A})$ is foliated by the stable foliation $\mathcal{F}^s$ whose leaves are the stable manifolds of the points in $\mathcal{A}$. The leaves of $\mathcal{F}$ are as differentiable as f but the transverse structure is just C^0. The stable bundle varies C^0 with f.

Let q be a saddle point of f having the same index as $\mathcal{A}$, and assume that $W^u(q) \cap W^s(\mathcal{A})$ is non-empty. Assume that there is an open subdisk $D \subset W^u(q)$ contained in $W^s(\mathcal{A})$ and such that, for a (local) orientation of $\mathcal{F}^s$ defined in the neighborhood of D, there are two points of D where D cuts $\mathcal{F}^s$ with opposite orientations.

The following lemma follows directly from Remark 3.1:

Lemme 3.2. *With he hypotheses above, $W^u(q)$ has a robust tangency with the stable manifold of $\mathcal{A}$.*

3.2 Example of C^1-robust tangencies

For building a robust tangency, one needs an hyperbolic set whose stable (or unstable) manifold has a larger dimension than the stable manifolds of each of its point. An easy example is given by the (non-trivial) hyperbolic attractors: the stable manifold of a nontrivial attractor is an open set, foliated by the (lower dimensional) stable manifolds of its points.

Consider a axiome A diffeomorphism φ of the 2 sphere S^2 whose non-wandering set consists in exactly a finite number of repelling fixed points α_i and a hyperbolic Plykin attractor $\mathcal{A}$. Removing a small disk in the basin of the repeling point α_0, one gets an attracting disk $\mathbb{D}^2$ for φ. Let λ be a upper bound of the unstable derivative of φ on the attractor $\mathcal{A}$ and on the finitely many repelling points, that is

$$\lambda > \sup\{|D^u\varphi(z|, z \in \mathcal{A}\} \cup \{\|Df(\alpha_i\|\}$$

One may assume that

- φ coincides with $z \mapsto \frac{z}{2}$ on the neighborhood of $\partial\mathbb{D}^2$;
- φ is isotopic to the homothety $z \mapsto \frac{z}{2}$ relatively to a neighborhood of $\partial\mathbb{D}^2$, meaning that the diffeomorphism coincides with the homothety all along the isotopy φ_t, where $\varphi_\varepsilon = \varphi$, $\varphi_{1-\varepsilon}\colon z \mapsto z2$ for every small ε.
- φ_t is a smooth isotopy: one assume that $(x,t) \mapsto (\varphi_t(x), t)$ is a diffeomorphisms.

Multiply this 2-disk with a transverse expansion: one get a diffeomorphism on $\mathbb{D}^2 \times \mathbb{R}$. One denotes by $\Psi\colon \mathbb{D}^2 \times \mathbb{R} \to \mathbb{D}^2 \times \mathbb{R}$ the diffeomorphism defined by $(z,t) \mapsto \varphi_{\inf\{|t|,1\}}(z), \lambda t)$.

Notice that

- Ψ coincides with the linear saddle map $(x, y, t) \mapsto (\frac{x}{2}, \frac{y}{2}, \lambda t)$ out of a compact set contained in $int(\mathbb{D}^2) \times (-1, 1)$
- the disk $\mathbb{D}^2 \times \{0\}$ is an invariant normally hyperbolic disk. As a consequence it persist by C^1-small perturbation of ψ.

Consider a diffeomorphism f_0 of $\mathbb{R}^3$, and assume that f_0 has a saddle point p_0 such that f_0 coincides with the linear map $(x,y,t) \mapsto (\frac{x}{2}, \frac{y}{2}, \lambda t)$ in small linearizing coordinates (x,y,t) around p_0.

One denotes by f_ε the diffeomorphism such that

- f_ε coincides with f_0 out of $\{\sqrt{x^2+y^2} < \varepsilon, |t| < \varepsilon\}$
- f_ε coincides with $(x,y,t) \mapsto \varepsilon\Psi(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\varepsilon})$ on $\{\sqrt{x^2+y^2} < \varepsilon, |t| < \varepsilon\}$.

We denote by $\mathcal{A}_\varepsilon$ the hyperbolic basic set of f_ε corresponding to $\mathcal{A}$.

Theorem 3.1. *Assume now that p_0 has a transverse homoclinic intersection. Let q be a periodic point homoclinically related with p. Then there is $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ one has*

- *f_ε has a robust cycle relating the hyperbolic basic set $\mathcal{A}_\varepsilon$ and the hyperbolic saddle q*
- *f_ε has a robust tangency associated to $\mathcal{A}_\varepsilon$.*

This fact is the idea for Abraham Smale examples [AS], Simon examples [Si], and the argument is more explicit in Asaoka recent work [As] and in [BLY].

3.3 A local mecanisms for C^1-robust tangencies

This chapter is dedicated to the results in [BD$_4$]. One first defines a special kind of hyperbolic basic set, called *blender horsehoe*, and which will substitute the set $\mathcal{A}_\varepsilon$ (Plikin attractor multiply by a tranverse expansion) of the examples f_ε in the previous section. As $\mathcal{A}_\varepsilon$, its a hyperbolic basic set whose stable manifold has larger *dimension* than its index, that is the dimension of the stable manifold of the hyperbolic set is larger than the stable manifold of each point in the basic set.

3.3.1 Blender horseshoes

A *horseshoe* of a diffeomorphism f is a hyperbolic basic set admiting a Markov partition consisting in precisely 1 rectangle R, and such that the intersection $f(R) \cap R$ contains precisely 2 connected components.

A *cu-blender horseshoe* is a horseshoe with additional properties. More precisely, a hyperbolic basic set Λ of a diffeomorphism f of a 3-manifold M is a blender horseshoe if

1. there is a cube $C \simeq [-1,1]^3$ embedded in M such that Λ is contained in the interior of C and is the maximal invariant set in C.
2. Λ is hyperbolic, C is a Markov partition of Λ and $C \cap f^{-1}(C)$ consists in two connected components A and B, which are disjoint from $\partial^u(C) = [-1,1] \times \partial([-1,1]^2)$; furthermore, $f(A)$ and $f(B)$ are disjoint from $\partial^s(C) = \{-1,1\} \times [-1,1]^2$.
3. In particular, Λ contains exactly two fixed points $p \in A$ and $q \in B$. One call local stable manifolds and one denotes by $W^s_{loc}(p)$ and $W^s_{loc}(q)$ the connected component of $W^s(p) \cap C$ and $W^s(q) \cap C$ containing p and q. The local stable manifolds of p and q are segments joining the two faces of the stable boundary of the cube C, that is, joining $\{-1\} \times [-1,1]^2$ to $\{1\} \times [-1,1]^2$.
4. there is a splitting $E^s \oplus E^c \oplus E^u$ defined on C, with the following properties:
   (a) $dim E^s = dim E^c = dim E^u = 1$
   (b) the splitting is Df invariant (that is, for $x \in A \cup B$ the splliting at $f(x)$ is the image by $Df - x$) of the splitting at x.
   (c) Df contracts uniformly the vectors in E^s, expands uniformly the vectors in $E^c \oplus E^u$ and expands uniformly stronger the vectors in E^u than in E^c.
5. there is $\alpha > 0$ such that the cone-field $\mathcal{C}^u_\alpha(x) = \{(v_1, v_2, v_3) \in T_xM, \sqrt{(v_1)^2 + (v_2)^2} \le \alpha|v_3|\}$ is strictly invariant by $Df(x)$, $x \in A \cup B$, that is $Df(\mathcal{C}^u_\alpha(x) \subset \mathcal{C}^u_{\alpha'}(f(x))$ with $\alpha' < \alpha$.

6. for every vector $v \in \mathcal{C}^u_\alpha$ the plane generated by v and $\frac{\partial}{\partial v_2}$ is transverse to E^s.

7. A *vertical segment* σ is a segment tangent to $\mathcal{C}^u_\alpha$ and joining $[-1,1]^2 \times \{-1\}$ to $[-1,1]^2 \times \{-1\}$. By the item above, the plane obtained from σ by considering the union of translated segments $\sigma + (0,t,0)$, $t \in \mathbb{R}$, cuts the local stable manifolds $W^s_{loc}(p)$ and $W^s_{loc}(q)$ each in exatly 1 point.

   Hence, a segment disjoint from $W^s_{loc}(p)$ is at the right or at the left of $W^s_{loc}(p)$: σ is at the left of $W^s_{loc}(p)$ is that is $t > 0$ with $\sigma + (0,t,0) \cap W^s_{loc}(p) \neq \emptyset$.

   One assume that

   (a) any vertical segment σ intersecting $W^s_{loc}(p) \cup W^s_{loc}(q)$ is disjoint from the left and right faces of C that is $\partial^{left}(C) = [-1,1] \times \{-1\} \times [-1,1]$ and from $\partial^{right}(C) = [-1,1] \times \{1\} \times [-1,1]$.

   (b) any vertical segment meeting $W^s_{loc}(p)$ is disjoint and at the left from $W^s_{loc}(q)$.

8. According to the previous item, a vertical segment σ has 5 possible positions:

   - at the left of $W^s_{loc}(p)$ (hence also at the left of $W^s_{loc}(q)$); for being short, we say that σ is at the left;
   - intersecting $W^s_{loc}(p)$ (hence at the left of $W^s_{loc}(q)$)
   - at the right of $W^s_{loc}(p)$ and at the lef of $W^s_{loc}(q)$; in that case we will say that σ is inbetween ($W^s_{loc}(p)$ and $W^s_{loc}(q)$);
   - intersecting $W^s_{loc}(q)$ (hence at the right of $W^s_{loc}(p)$)
   - at the right of $W^s_{loc}(q)$ (hence also at the right of $W^s_{loc}(p)$; one says that σ is at the right.

   One assume that

   (a) if σ is at the right (resp. left) of $W^s(p)$ and if $f_A(\sigma)$ is a vertical segment, then $f_A(\sigma)$ is at the right (resp. left) of $W^s_{loc}(p)$;

   (b) if σ is at the right (resp. left) of $W^s(q)$ and if $f_B(\sigma)$ is a vertical segment, then $f_B(\sigma)$ is at the right (resp. left) of $W^s_{loc}(q)$;

   (c) for every vertical segment σ inbetween, $f_A(\sigma)$ or $f_B(\sigma)$ is a vertical segment inbetween.

   (d) for every vertical segment σ through $W^s_{loc}(p)$, $f_B(\sigma)$ is not a vertical segment inbetween.

Remark 3.3. *Having a blender horseshoe is a C^1-open property: if C is the cube defining a blender horseshoe for f then there is a C^1 neighborhood $\mathcal{U}$ of f for which C defines a blender horseshoe for every $g \in \mathcal{U}$.*

3.3.2 example of blender horseshoe

Consider a diffeomorphism φ of $\mathbb{R}^2$ having a usual horseshoe in a rectangle R. Let a and b be the connected components of $R \cap \varphi^{-1}(R)$. One assume that the unstable derivative of f is uniformely larger than 2 on R

Let $f_{\lambda,s} \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the diffeomorphism such that

- $f_{\lambda,s}(x_1,x_2,x_3)$ is on the form $(\varphi(x_1,x_2), \psi_{x_1,x_2}(x_3))$,
- $f_{\lambda,s}(x_1,x_2,x_3) = (\varphi(x_1,x_2), \lambda x_3$ if $(x_1,x_2) \in a$
- $f_{\lambda,s}(x_1,x_2,x_3) = (\varphi(x_1,x_2), \lambda x_3 + s$ if $(x_1,x_2) \in b$

Then for every $\lambda \in (1,2)$ and every $t \neq 0$, $f_{\lambda,s}$ has a horseshoe blender in the cube $R \times [\frac{-2s}{\lambda-1}, \frac{2s}{\lambda-1}]$. (a complete proof of this fact was written in [BDV$_2$].

3.4 Blender horseshoe and robust tangencies

Let C be the cube of a blender horseshoe. A fold is a square $S\colon [0,1]^2 \to M$ where:

- $S = \bigcup_0^1 \sigma_t$ where $\sigma_t\colon [0,1] \to M, r \mapsto S(t,r)$ is a continuous family of vertical segments
- σ_0 and σ_1 cut $W^s_{loc}(p)$
- for every $t \in (0,1)$, σ_t is inbetween.

Remark 3.4. *If $\theta\colon N \to M$ is an immersed surface transverse to each sides of ∂C, tranverse to the local stable manifolds $W^s_{loc}(p)$ and $W^s_{loc}(q)$ containing a fold $S = \bigcup_0^1 \sigma_t$ with σ_t tangent to the smaller cone $\mathcal{C}^u_{\frac{\alpha}{2}}$, then there is a C^1 neighborhood $\mathcal{U}_N$ of (θ, N) and a C^1-neighborhood $\mathcal{U}_f$ of f such that, for every $N' \in \mathcal{U}_N$ and every $g \in \mathcal{U}_f$, the surface N' contains a fold for the blender horseshoe of g.*

Theorem 3.2. *If C is the cube defining a blender horseshoe Λ of f and $S = \bigcup_0^1 \sigma_t \subset C$ is a fold, then there is a point $x \in \Lambda$ such that $W^s(x)$ is tangent to S at some point $y \in W^s(x) \cap S$.*

The main step for proving the theorem is

Lemme 3.5. *$f(S) \cap C$ contains a fold S_1.*

proof of the Theorem assuming the Lemma : The lemma allows us to define by induction a sequence S_i of folds with $S_{i+1} \subset f(S_i) \subset f^i(S)$. Consider the decreasing sequence $\Sigma_i = f^{-i}(S_i)$. Then $\Sigma = \bigcap_i \Sigma_i$ is a non-empty compact set. Every point in Σ has all its positive iterates in C hence belongs to the stable manifold of Λ. Furthermore, every fold S_i contains a point x_i tangent to the vectorfield $\frac{\partial}{\partial x_1}$ which is contained in the stable cone $\mathcal{C}^s_\alpha$ which is invariant by negative iterates. Every accumulation point of the sequence of negative iterates $f^{-i}(x_i) \in \Sigma \subset S$ is a tangency point of S with $W^s(\Lambda)$. □

It remains to prove the lemma.

Proof : First assume that none of the $f_B(\sigma_t)$ is a vertical segment inbetween. Then, by assumptuion, for every $t \in (0,1)$ $f_A(\sigma_t)$ is a segment between. Then $f_A(\sigma_i,\, i = 0,1$ is a vertical segment through $W^s_{loc}(p)$. So $f_A(S)$ is a fold.

Now assume that there is t_1 such that $f_B(\sigma_{t_1})$ is inbetween. Let $0 \le t_0 < t_1 < t_2 \le 1$ such that $f_B(\sigma_t)$ is inbetween for every $t \in (t_0, t_1)$ and such that (t_0, t_1) is the largest interval with this property. Then σ_{t_0} and σ_{t_2} are through $W^s_{loc}(p)$. Hence $S_1 = \bigcup_{t_0}^{t_2} f_B(\sigma_t)$ is a fold, ending the proof. □

3.5 abundance of C^1-robust tangencies

If p is a saddle-node point and if the strong stable and strong unstable manifolds of p have a homoclinic intersection, then a small perturbation of f build a blender horseshoe for some iterate of f, containing the point p.

If p and q are hyperbolic periodic saddle points such that $index(p) + 1 = index(q)$ and if there is a heterodimensional cycle associated ot p and q, then a small perturbation of f creates a saddle node point r such that $W^u(p) \cap W^s(r) \neq \emptyset \neq W^s(q) \cap W^u(r)$. Hence, if p and q belong robustly to the same chain recurrence class, one gets a blender horseshoe (Λ, C) of the same index as p, in the chain recurrence class of p; more precisely, the blender horseshoe Λ and p are contained in a larger basic set K_f.

Recall that, according to [Ab, BC], for generic diffeomorphisms, if two periodic points belongs to the came chain recurrence class, they belong robustly to the same chain recurrence class.

Now, if the stable unstable splitting is not dominated along the periodic orbits homoclinically related with p, then [Go] allows to create a homoclinic tangency associated to p. Iterating a small rectangle of $W^u(p)$ around the tangency point, and performing a small perturbation, one build a (robust) fold contained in $W^u(p) \cap C$. Hence one gets that K_f has a robust tangency.

A standard argument of genericity allow nos to prove Theorem 2.1 and Corollary 2.1.

4 C^2-robust tangencies in dimension 2

4.1 Homoclinic tangency and intersection of Cantor sets

In this section, I will just explain the rough idea that many people knows, just for justifying the fact that the heart of the study consists in analysing dynamical Cantor sets in dimension 1. See [PT] for a more serious explaination.

Let f be a diffeomorphisms of a compact surface, having a hyperbolic basic set K of saddle type. Then K admits a generating Markov partition by disjoint rectangles.

The local stable manifold of K (i.e. the set of point whose positive iterates remain in the rectangles of the Markov partition = the intersection of the negative iterates of the union of the rectangle of the Markov partition) is homeomorphic to the product of a Cantor set by a segment; the leaves form a continuous family of segments which are as smooth as f; furthermore, if f is of class C^2, this local stable manifold $W^s_0(K)$ may be embedded in a C^1-foliation $\mathcal{F}^s$.

In the same way, local unstable manifold $W^u_0 K$ is homeomorphic to the product of a Cantor set by a segment as smooth as f; if f is of class C^2, this local stable manifold $W^s_0(K)$ may be embedded in a C^1-foliation $\mathcal{F}^u$.

Let us denote $W^u_n(K) = f^n(W^u_0(K)$; it is a larger local stable manifold of K. In the same way one defines $W^s_n(K) = f^{-n}(W^s_0(K))$.

Assume now that, at some place, a leaf of the unstable foliation $\mathcal{F}^u$ makes a quadratic tangency with a leaf of $\mathcal{F}^s$ at a point x. Then, as the leaves are C^2 and depends C^2-continuously on the point, there is a neighborhood U_x of x where the tangency point between $\mathcal{F}^s$ and $\mathcal{F}^u$ form a C^0 curve γ topologically transverse to both foliations.

Now, γ is a segment and $\gamma \cap W^s_n(K)$ is a Cantor set K^s and $\gamma \cap W^u_n(K)$ is a Cantor set K^u.

There is a homoclinic tangency associated to K for the local stable manifolds if $K^s \cap K^u \neq \emptyset$.

4.2 The geometric part of Newhouse argument

4.2.1 Thickness of a Cantor set

Given a Cantor set $K \subset \mathbb{R}$ a *gap* of K is a connected component of $\mathbb{R} \setminus K$.

Definition 4.1.
- *Given a gap I let $t(I)$ denote* $\inf \frac{\ell(U)}{\ell(I)}$ *where U is the smallest interval joining I to a gap larger or equal to I.*
- *U will be called an interval which is* adjacent to the gap *I: each gap has two adjacent intervals.*
- *One denote*
$$t(K) = \inf_{I \text{ gap of } K} t(I) \in [0, +\infty),$$
the thickness of K.

Remark 4.2. *Given a gap I, the gaps contained in its adjacent intervals are strctly smaller, by definition of adjacent intervals.*

4.2.2 Thickness and intersection of Cantor sets

Theorem 4.1. *If K and L are Cantor sets such that $t(K)t(L) > 1$ then either K is contained in a gap of L or conversely L is contained in a gap of K or else $K \cap L \neq \emptyset$.*

We will argue by contradiction, assuming that K and L are disjoint, but there is $a, b \in K$ and $c, d \in L$ such that $a < c < b < d$ or $c < a < d < b$; one says thatthe intervals $[a, b]$ and $[c, d]$ are *linked*.

As K and L are disjoint compact set, one easily proves:

Lemme 4.3. *The set of $(a, c, b, d) \in K^2 \times L^2$ such that $[a, b]$ and $[c, d]$ are linked is a compact open subset of $K^2 \times L^2$ (disjoint from the diagonals)*

Assume that I (with extremities in K) and J (with extremities in L) are linked. Then I has exactly one extremity in J. This extremity belongs to a gap J_0 of L in J (because K and L are disjoint). Now J_0 has its extremities in L and I and J_0 are linked because I has exactly one extremity in J_0; hence one may repeat the argument: one extremity of J_0 belongs to a gap I_0 of K contained in J. We just proved :

Lemme 4.4. *Assume that I (with extremities in K) and J (with extremities in L) are linked. Then there is a gap $I_0 \subset I$ of K and a gap $J_0 \subset J$ of L such that I_0 and J_0 are linked.*

Given a pair of linked interval I, J one associates the pair $L(I,J) = (\inf\{\ell(I), \ell(J)\}, \sup\{\ell(I), \ell(J)\}) \in \mathbb{R}^2$. We denote by $<_{lex}$ the lexicographic order on $\mathbb{R}^2$. Putting together the two lemma, one easily gets:

Corollaire 4.5. *There is a gap I of K and a gap J of L such that the pair $L(I,J)$ realise the infimum for $<_{lex}$ of the $L(I',J')$ for every linked pair.*

We will conclude the proof of Theorem 4.1 by proving:

Lemme 4.6. *For every linked pair I, J such that I is a gap of K and J is a gap of L, there is a linked pair I_0, J_0 with $L(I_0, J_0) <_{lex} L(I,J)$.*

Proof : I has an extremity in J. Let U be the adjacent interval (for K) of I starting at this extremity. In particular one extremity of U in contained in the interior of J. If $\ell(U) \geq \ell(J)$ then J has an extremity in U, hence in a gap I_0 of K in U. Recall that the gap of K contained in U are strictly smaller that I: that is $\ell(I_0) < \ell(I)$ and the pair (I_0, J) is linked. So (I_0, J) is the announced pair.

In the same way, let V be the adjacent interval (for L) of J starting at the extremity of J in the interior of I. If $\ell(V) \geq \ell(I)$ then I has an extremity in V, hence in a gap J_0 of L in V with $\ell(J_0) < \ell(J)$ and the pair (I, J_0) is linked. So (I, J_0) is the announced pair.

It remains the case $\ell(U) < \ell(J)$ and $\ell(V) < \ell(I)$. That is $\frac{\ell(U)}{\ell(J)} < 1$ and $\frac{\ell(V)}{\ell(I)} < 1$. As a consequence

$$t(K)t(L) \leq \left(\frac{\ell(U)}{\ell(I)}\right)\left(\frac{\ell(V)}{\ell(J)}\right) = \left(\frac{\ell(V)}{\ell(I)}\right)\left(\frac{\ell(U)}{\ell(J)}\right) < 1.$$

This contradics the hypothesis $t(K)t(L) > 1$.

□

4.3 Dynamical cantor sets in $\mathbb{R}$

The thickness provides a geometric criterium implying that two Cantor sets allways meet. How can we ensure that the Cantor set we are considering are thick enough? This geometric properties comes from how the Cantor sets are generated. They are not any Cantor set, they are dynamical Cantor sets.

4.3.1 Definition: expanding map, filtrating set

We consider C^1-maps ψ defined on a compact set $U \subset$ which is a finite union of compact segments $U_1, \dots U_r$. One says that ψ is an expanding map if the derivative is larger than 1 in modulus:

$$\forall x \in U, |\psi'(x)| > 1.$$

In particular the expanding map ψ is a diffeomorphism in restriction to every connected component of U.

One says that U is a *filtrating set* for the expanding map ψ if for every $i, j \in \{1, \dots, r\}$ one has

$$U_i \cap f(U_j) \neq \emptyset \Longrightarrow U_i \subset Int(f(U_j))$$

We denote by $\Lambda(\psi, U)$ the maximal invariant set of ψ in U:

$$\Lambda(\psi, U) = \bigcup_{n \in \mathbb{N}} \psi^{-n}(U).$$

In other words, $\Lambda(\psi, U)$ is the set of points whose positive orbits by ψ is allways defined, because it remains in the domain U of ψ.

Notice that, by construction, $\Lambda(\psi, U)$ is a hyperbolic set of ψ, and the classical hyperbolic theory implies:

Theorem 4.2. *The restriction of ψ to $\Lambda(\psi, U)$ is conjugated to the one sided finite type subshift associated to the incidence matrix associated to ψ and to the connected components of U*

Proof : Just consider the itineraries in the connecting components of U. □

Proposition 4.7. *The set $\Lambda(\psi, U)$ admits a basis of filtrating neighborhoods.*

Proof : Just consider the connecting components of the finite intersections $\bigcap_{n=0}^{m} \psi^{-n}(U)$. □

Proposition 4.8. *The set $\Lambda(\psi, U)$ has empty interior.*

Proof : The length of the positive iterates $\psi^n(C)$ of a connected component C of $\Lambda(\psi, U)$ is increasing exponentiall with $n > 0$ but remains bounded, hence is 0. □

One says that two pairs (ψ, U) and (φ, V) of expanding maps defined on a filtrating set *defines the same dynamical set* K if:

- $K = \Lambda(\psi, U) = \Lambda(\varphi, V)$
- K admits a (filtrating) neighborhood W such that the restriction of ψ and φ to W coincide.

Hence, a *dynamical set* K is a germ of an expanding map ψ at the neighborhood of the maximal invariant set $\Lambda(\psi, U)$ in a filtrating neighborhood U.

4.3.2 The C^r-topology on the dynamical sets

Consider a dynamical set $(K, [\psi])$, where $[\psi]$ is a germ at K of a C^r-expanding map. A C^r-neighborhood of (K, ψ) is given by:

- a realization ψ of the germ $[\psi]$
- a fitrating neighborhood U of K for ψ
- a C^r neighborhood $\mathcal{U}$ of the restriction of ψ to U, small enough so that U is a filtrating st for every $\varphi \in \mathcal{U}$.

4.3.3 Thickness of C^2-dynamical Cantor set

Theorem 4.3. *The thickness $t(K)$ depends continuously on K in the C^2 topology. Furthermore, for any C^2-dynamical Cantor set K, the thickness $t(K)$ does not vanish.*

This comes from a (now classical) distorsion lemma:

Lemme 4.9. *Let (K, ψ, U) be a C^2 dynamical Cantor set. Then there is a constant $C > 0$ such that, for every $n \in \mathbb{N}$ and every interval $I \subset U$) on which ψ is defined, for every $x, y \in I$ one has*

$$|\log D\psi^n(x) - \log D\psi^n(y)| < C.$$

Proof : As ψ is uniformly expanding, one gets that $\sum_0^n d(\psi^i(x), \psi^i(y)$ is uniformly bounded, independently to n, I and $x, y \in I$. Now using the fact that $\log D\psi$ is a C^1-map (hence is Lipschitz) one gets that $\sum_0^{n-1} |\log D\psi(\psi^i(x)) - \log D\psi(\psi^i(y))|$ is uniformly bounded, and one concludes by noticing that $|\log D\psi^n(x) - \log D\psi^n(y)| = |\sum_0^{n-1} \log D\psi(\psi^i(x)) - \log D\psi(\psi^i(y))|$ □

so one gets:

Theorem 4.4. *Let K_0,L_0 be two C^2-dynamical Cantor sets such that $t(K_0)t(L_0) > 1$, and such that they admits a pair of linked inttervals. Then, there are C^2-neighborhoods $\mathcal{U}$,$\mathcal{V}$ of K_0 and L_0 such that for every $K \in \mathcal{U}$ and $L \in \mathcal{V}$ one has*

$$K \cap L \neq \emptyset$$

5 No C^1-robust tangency in dimension 2

As we have seen, Newhouse argument uses the thickness which is a global geometrical invariant. This geometric invariant has many very bad properties: the thickness of a subset $L \subset K$ may be larger than the thickness of K. Worst: the thickness of the union $K = K_1 \cup K_2$ of two Cantor sets K_1 and K_2 may be arbitrarily small, independently on $t(K_1)$ and $t(K_2)$.

Looking now to C^1-perturbation of dynamical Cantor set, the first natural question was to understand if Newhouse argument holds in that topology. This as been solved by Raul Ures in his thesis, published in [U].

5.1 Ures and Moreira's result

5.1.1 Ures result: genericity of 0 thickness

Theorem 5.1. *Given any C^1-dynamical Cantor set K and given any $\varepsilon > 0$ there is are dynamical Cantor sets K' arbitrarily C^1-close to K, such that $t(K') < \varepsilon$.*

As $t(K)$ varies upper-semi continuously with K (because $t()$ is an infimum of continuous fonctions) one gets:

Corollaire 5.1. *There is a C^1-residual subset $\mathcal{R}$ of the set of C^1-dynamical Cantor sets such that any $K \in \mathcal{R}$ as thickness equal to 0.*

5.1.2 Separating two dynamical Cantor sets: the result of Carlos Gustavo Moreira

Theorem 5.2. *Let $(K, [\psi])$ and $(L, [\varphi])$ be two dynamical sets. Then, every C^1-neighborhoods $\mathcal{V}_K$ and $\mathcal{V}_L$ of $(K, [\psi])$ and $(L, [\varphi])$ contain a pair $((K', [\psi']), (L', [\varphi']))$ such that $K' \cap L' = \emptyset$.*

Firts remark that K and L either are countable or contain a Cantor set K_∞ and L_∞. It contains a Cantor set if and only if the incidence matrix have a eigenvalue of modulus larger than 1.

Recal that an incidence matrix A is called *mixing* or *indecomponible* if there is a power A^k such that all the entries are strictly positive. In that case, one says that the corresponding dynamical Cantor set is a mixing dynamical Cantor set.

The theorem is a consequence of the same statement for mixing dynamical Cantor sets.

Theorem 5.3. *Let $(K, [\psi])$ and $(L, [\varphi])$ be two mixing dynamical Cantor sets. Then, every C^1-neighborhoods $\mathcal{V}_K$ and $\mathcal{V}_L$ of $(K, [\psi])$ and $(L, [\varphi])$ contain a pair $((K', [\psi']), (L', [\varphi']))$ such that $K' \cap L' = \emptyset$.*

5.2 Dynamical cantor sets in $\mathbb{R}$

5.2.1 Markov partitions

A Cantor set $K \subset \mathbb{R}$ is called a C^r-dynamical Cantor set if

- There are disjoint compact segments $I_1, \ldots, I_r \subset \mathbb{R}$, $r \in \mathbb{N}$, ordered in an increasing way in $\mathbb{R}$ such that
  - $K \subset \bigcup_{j=1}^r I_j$;
  - for every $j \in \{1, \ldots, r\}$ the boundary of I_j is contained in K:

$$\partial I_j \subset K$$

- there is a compact neighborhood U of $\bigcup_{j=1}^r I_j$ and a C^1-map $\psi \colon U \to \mathbb{R}$ with the following properties
  - $\psi(U)$ contains U in its interior.

– ψ is uniform dilatation: $\psi'(x) > 1$ for all $x \in U$. In particular, the restriction of ψ to each connected component of U is a C^1-diffeomorphism.

– for every $i \in \{1, \ldots, r\}$ there is a $j \leq k \in \{1, \ldots, r\}$ such that $\psi(I_i)$ is the convex hull of $I_j \cup I_k$ (as the I_j are indexed in an increasing way, this convex hull contains I_ℓ if and only if $j \leq \ell \leq k$). In other words, the segments I_i form a Markov partition.

– K is the maximal invariant set in U:

$$K = \bigcap_{n \in \mathbb{N}} \psi^{-n}(U)$$

– The markov partition is mixing: for every $j \in \{1, \ldots, r\}$ there is n such that $\psi^n(I_j)$ contains all the I_k. In other words,

$$\psi^n(I_j \cap K) = K.$$

This is equivalent to the fact tha tthe incidence patrix of the Markov partition has a power whose entries are all > 0.

We say that $\{I_1, I_2, \ldots, I_r\}$ is a *Markov partition for* K, and that K *is defined by* ψ, and U is an isolating neighborhood of K. We say that K is a C^r-dynamical Cantor set if it is defined by a C^r-expanding map ψ.

5.2.2 C^r-Perturbations of a dynamical Cantor set

Remark 5.2. *Let $(K, \psi, \{I_i\}, U)$ be a dynamical Cantor set endowed with a Markov partition, a defining expanding map and an isolating neighborhood.*

Then every maps ψ': C^r-close to ψ defines a new dynamical cantor set $(K', \psi', \{I'_i\}, U)$ endowed with a markov partition I'_i whose end points vary continuously and whose incidence matrix is equal to the one of K. One says that $(K', \psi', \{I'_i\}, U)$ is C^r-close to $(K, \psi, \{I_i\}, U)$

This topology on the set of triple $(K, \{I_i\}, \psi)$ does not depend on the choice of the isolating neighborhood U. But the C^r-distance defining this topology depends on U.

We denote by $\lambda((K, \{I_i\}, \psi)$, or shortly $\lambda(K)$ the bound of the derivative of ψ on the Markov partition:

$$\lambda(K) = \max \left\{ |\psi'(x)|, x \in \bigcup_{j=1}^{r} I_j \right\}.$$

Notice that, once fixed the Markov partition $\{I_j\}$, the bound $\lambda(K)$ varies continuously with K for the C^1-topology.

5.2.3 Affine dynamical Cantor set

A dynamical Cantor set (K, ψ, U) is *locally affine* if there is a neighborhood U' of K such that the restriction of ψ to every connecting component of U' is affine.

Lemme 5.3. *The set of locally affine dynamical Cantor sets is C^1-dense in the set of dynamical Cantor sets.*

Every locally affine Cantor set (K, ψ, U) admits a Markov partition $\{I_i\}$ such that ψ is *affine* in a neighborhood of every I_i. One says that the dynamical Cantor set $(K, \{I_i\}, \psi, U)$ is affine.

5.3 Enlarging gaps: a C^1-perturbation lemma

5.3.1 Intervals and gaps

Let $(K, U, \psi, \{I_i\}_{i\in\{1,\dots,r\}}$ be a dynamical Cantor set endowedd with a Markov partition.

An *interval of the construction* is a connecting component of $\psi^{-n}(I_i)$, $n \in \mathbb{N}$ and $i \in \{1, \dots, r\}$. The interval of the construction are compact intervals but they intersect K along open and close subset (clopen subsets). For every $n \in \mathbb{N}$, an *interval of generation* n is a connecting component of $\psi^{-n}(I_i)$, $i \in \{1, \dots, r\}$. Notice that a same interval I of the construction may be of different generations.

We denote by $\mathcal{I}$ the set of intervals of the construction. We denote by $\mathcal{I}_n$ the set of intervals of generation n.

A gap is a connected component of $\mathbb{R} \setminus K$. We stratify the set of gaps in generation:

- a generation 0 gap is a connected component of $\mathbb{R} \setminus \mathcal{I}$ where $\mathbb{I} = \bigcup_1^r I_i$.
- a generation 1 gap is a connected component of $\mathbb{I} \setminus \psi^{-1}(\mathbb{I})$
- A *gap* u *of generation* n is a connected component u of $\psi^{n-1}(\mathbb{I}) \setminus \psi^{-(n)}(\mathbb{I})$. A generation n gap is contained in a generation $n-1$ I interval of the construction: u is an open interval contains in I so that $\psi^{n-1}(u)$ is well defined but $\psi^n(u)$ is an open interval disjoint from the I_j and whose extremities are endpoints of the I_j.

Remark 5.4. *1. If I is an interval of generation n, then ψ^{n+1} is defined and expanding on I.*

2. two different intervals of the construction are either disjoint or one contains the other.

3. An interval of generation n may not have any gaps of generation $n+1$.

4. two different gaps are always disjoint.

5.3.2 Ratios gaps/intervals

Every interval of the contruction I of generation n shares its extremities with two gaps u_I^- and u_I^+, whose generation is at most n.

we denote $A(I) = \inf\{\frac{\ell(u_I^-)}{\ell(I)}, \frac{\ell(u_I^+)}{\ell(I)}\} \in (0, +\infty)$, and $A(K) = \inf_{I\in\mathcal{I}} A(I)\}$.

Remark 5.5. *If $(K, \psi, \mathcal{I})$ is a affine dynamical Cantor set, then*

$$A(K) = \inf_{\{I\in\mathcal{I}\}}(A(I)) > 0.$$

.

A classical argument using the control of distorsion shows

Lemme 5.6. *If K is a C^2-dynamical Cantor set, then $A(K) > 0$.*

To every interval I of the construction, we will assign a specific gap $u_I \subset I$, as follows. We consider the time n such that $\psi^n(I)$ is defined and equal to some of the 0-generation intervals I_k but $\psi^{n+1}(I)$ contains the convex hull of exactly $I_i, I_{i+1}, \dots, I_j$ for $i < j \in \{1, \dots r\}$. In other words, n is the largest n for which I is a n-generation interval. Then

$$u_I = \psi^{-(n+1)}(]\sup I_i, \inf I_{i+1}[).$$

In other words, u_I is a gap of the smallest generation n contained in I. The gaps of these generations are disjoint, hence naturally ordered. Then u_i is the first or the last of these intervals, according to the sign (> 0 or < 0) of the derivative ψ^{n+1} on I.

Next remark will allow us to perform perturbations, keeping the control of the gaps and intervals:

Remark 5.7. *If I is an interval of largest generation n, and J is an interval of generation larger than $n+1$. Then the positive orbit of u_I, u_I^+ and u_I^- are disjoint from the interior of J: in particular, any perturbation of ψ supported in J does not change I, u_I, u_I^+ and u_I^-.*

For every interval of the construction I one defines $a(I) = \frac{|u_I|}{|I|}$ and we define

$$a(K, \{I_i\}_{i\in\{1,\dots,r\}}) = \inf\{a(I), I \text{ interval of the construction}\}.$$

Remark 5.8. *1. If K is an affine Cantor set, then $a(K)$ is given by the 0-generation intervals; in particular $a(K) > 0$. If K is locally affine, then it is affine on the n-generation intervals for some $n > 0$; as a consequence $a(K) > 0$*

2. *there are C^1-dynamical Cantor sets with $a(K) = 0$. That is the case for the dynamical Cantor sets with positive measure built by Mañé.*

3. *the fact that $a(K) > 0$ or $a(K) = 0$ does not depends on the choice of the Markov partition, but just on $(K, \psi|_K)$.*

Lemme 5.9. *Il K is a C^r-dynamical Cantor set with $r > 1$, then $a(K) > 0$. Furthermore, $a(K)$ depends continously on K in the C^r-topology*

Proof : That is a typical argument of control of the distorsion for expanding maps: there is C such that, for every interval ψ of generation n,

$$\max\{\frac{(\psi^n)'(x)}{(\psi^n)'(y)}, x, y \in I < C.$$

The conclusion follows easily. □

Lemme 5.10. *Let $(K, \psi, \mathcal{I})$ be a dynamical Cantor set endowed with a Markov partition $\mathcal{I}$. Let $\mathcal{I}_n$ denote the set of n-generation intervals, for $n \in \mathbb{N}$. Let $(K', \psi', \mathcal{I})$ be a dynamical Cantor set such that the image by ψ' of any interval $J \in \mathcal{I}_n$ is precisely $\psi(J)$ and $\psi' \colon J \to \psi(J)$ is the unique affine map such that $\psi^{-1}\psi' \colon J \to J$ is increasing.*

Then

- *$a(K') \geq a(K)$ and $A(K') \geq A(K)$;*
- *for large n, K' maybe chosen arbitrarily C^1 close to K.*

Proof : The unique difficult point is the control of $a(K')$ and $A(K')$. Let us explain it.

Let B_k denote the boundary of the union of all n-generation interval. Notice that $B_k \subset B_{k+1}$. The hypothesis implies that $\psi = \psi'$ on B_n. Notice that $\psi(B_n) = B_{n-1}$.

As ψ' is affine on $\mathcal{I}_n$ $a(K')$ is determined by the ratios $a(I)$ for I intervals whose largest generation number is less or equal to $n-1$ (because, if I is a m-generation interval for $m \geq n$ then $a(I, \psi') = a(\psi'(I), \psi')$).

Then it remains to remark that if the largest generation number of I is smaller than $n-1$ then the boundary of $u_{I,\psi}$ belongs to B_n. So the positive orbit for ψ and ψ' of the extremities of $U_{I,\psi}$ coincide. This implies

$$u_{I,\psi} = u_{I,\psi'}$$

thus $a(I, \psi) = a(I, \psi')$.

For $A(K')$ just notice that

- One did not change $A(I)$ for the generation $\leq n$ intervals.
- For the higher generation interval, one has $A(I) \geq A(\psi(I))$, concluding.

□

5.3.3 Proof of Ures's theorem: opening a large gap in one interval of the construction

Let K, U, Ψ be a dynamical Cantor set. Fix some $\eta > 0$. We want to perform η C^1-small perturbation in order that $t(K') < \varepsilon$. As the locally affine Cantor sets are dense, we may assume that K is affine, and we fix a Markov partition $\mathcal{I}$ on which Ψ is affine.

We will now perform a perturbation whch will enlarge a gap, producing a very small thickness. As we want to make an arbitrarilly small perturbation, we will spread this pertubation along the time, that is we will perfor the perturbation along the iterates of an interval. For avoiding interactions between the perturbations on different iterates of the interval, we need that the intervall remain disjoint from itself during an arbitrarily large time

Lemme 5.11. *For any $n > 0$ there is $k > n$ and a component I of $\psi^{-k}(\mathcal{I})$ such that $I, \psi(I), \ldots, \psi^n(I)$ are pairwize dijoint.*

Fix $\delta \in (0, 1)$ such that any map ψ' such that $\frac{D\psi'}{D\psi} \in [1 - \delta, \frac{1}{1-\delta}]$ is an η-C^1 small perturbation of ψ (It is enough to chose δ such that $\frac{\delta}{1-\delta}\lambda < \eta$ (where λ is a bound of $D\psi$).

An easy calculation allows to verify:

Lemme 5.12. *For every $\varepsilon > 0$ there is $n(\varepsilon)$ with the following property Let $x < y \in (0, 1)$ be such that $y - x \geq a(K)$. There is a sequence of diffeomorphisms $\theta_1, \ldots, \theta_n$ of $[0, 1]$ with the following properties:*

- $D\theta \in [1 - \delta, \frac{1}{1-\delta}]$
- *let x_i, y_i defined by induction as $x_n, y_n = x, y$ and $\theta_i(x_{i-1}) = x_i$ and $\theta_i(y_{i-1}) = y_i$, then θ_i is affine on $[0, x_{i-1}]$ and on $[y_{i-1}, 1]$*
-
$$y_0 - x_0 > 1 - \frac{\varepsilon}{\max\{1, A(K)\}}$$

Proof : I did not make the calculation but Gugu [2] wrote that $n(\varepsilon)$ is more or less $\frac{-\log\varepsilon}{a(K)\varepsilon}$. In fact it is enough to get the existence of $n(\varepsilon)$, which is easy. □

Let $\xi\colon I \to [0, 1]$ be the unique increasing affine map. It send the prescribe gap u_I on an interval $[x, y] \subset (0, 1)$ with $y - x \geq a(K)$. We apply the lemma to this value of x, and y.

We consider the perturbation L, ϕ of K, ψ defined as follows:

- $\phi = \psi$ out of (an arbitrarily small neighborhood of) $\bigcup_0^n \psi^i(I)$
- The restriction $\phi\colon \psi^{i-1}(I) \to \psi^i(I)$ is $\psi^i\xi^{-1}\theta_i\xi\psi^{1-i}$
- (we have just to define ϕ on the adjacent gap to $\psi^i(I)$: it consists in smoothing ϕ with ψ.

Then L is a η perturbation of K.

Furthermore, the gap $u_{\psi^n(I)}$ remained unchanged by this perturbation (see remark 5.7). Now the prescribe gap $\tilde{u}_I$ is the preimage by ϕ^n of the gap $u_{\psi^n(I)}$. One deduces that the new gap satisfies

$$\frac{\ell(\tilde{u}_I)}{\ell(I)} \geq 1 - \varepsilon \min\{1, A(K)\}$$

The perturbation did'nt change the adjacent gaps. Now, one of the component J of $I \setminus \tilde{u}_I$ as a length bounded by $\frac{1}{2}\varepsilon\ell(I)\min\{1, A(K)\}$ and is bounded by a gap of size larger than $A(K)\ell(I)$ and by $\tilde{u}_I$. Then (for $\varepsilon < \frac{1}{2}$) one gets that $t(L) < \varepsilon$, ending the proof of Ures theorem.

[2] Gugu = Carlos Gustavo Moreira

5.3.4 A C^1-perturbation lemma: opening many large gaps

This section explains the well known idea, already used by Ures in [U], that small C^1-perturbation allow to enlarge a given gap. However one would like to open many gaps. The problem is that enlarging some gaps can shrink the nearby gaps. Hence it is natural to get that one may enlarge the gaps if one has a long time wandering set of intervals. Let us formalize this simple idea.

Given $n \in \mathbb{N}$ one says that a compact subset $X \in \bigcup_1^r I_i$ is *n-wandering* if

- X is contained in the definition domain of ψ^n;
- for every $0 \le i < j \le n$ the iterates $\psi^i(X)$ and $\psi^j(X)$ are disjoint.

For every compact subset $X \subset K$ and every $n > 0$ we denote by $\mathcal{V}_n(X)$ the union of the n-generation interval intersecting X. Notice that $\mathcal{V}_n(X)$ is a neighborhood of X. Furthermore, the family $\{\mathcal{V}_n(X)\}_{n\in\mathbb{N}}$ is a base of neighborhood of X.

One easily verifies:

Lemme 5.13. *For every $n \in \mathbb{N}$ and every n-wandering set $X \subset K$ there is $m \in \mathbb{N}$ such that $\mathcal{V}_m(X)$ is n-wandering.*

As a direct coroallary of Lemma 5.10 is:

Corollaire 5.14. *Let $(K, \psi, \mathcal{I})$ be a dynamical Cantor set endowed with a Markov partition $\mathcal{I} = \{I_i\}$, $n > 0$ and $X \subset K$ a n-wandering set. Given every $\varepsilon > 0$ there is m and a dynamical Cantor set $(K', \psi', \mathcal{I}')$ such that*

- *the C^1-distance between $(K, \psi, \mathcal{I})$ and $(K', \psi', \mathcal{I}')$ is less that ε*
- *$\mathcal{V}_m(X)$ is n-wandering*
- *let $\mathcal{V}'_m(X')$ denote the continuation of $\mathcal{V}_m(X)$; its a n-wandering set for ψ'. Then the restriction of ψ' each component of $\bigcup_{j=0}^{n-1}(\psi')^j(\mathcal{V}'_m(X'))$ is affine.*
- *$a(K') \ge a(K)$ and $A(K) \ge A(K')$;*

Proof : □

We can now state the perturbation lemma.

Proposition 5.15. *Let $(K, U, \psi, \mathcal{I})$ be a C^2-dynamical Cantor set and denote $c(K) = 2\frac{\lambda(K)}{a(K)}$. Given $\varepsilon > 0$ one denotes $n_\varepsilon = \left\lceil c(K)\frac{log\varepsilon^{-1}}{\varepsilon}\right\rceil$. Then given any n_ε-wandering compact subset $X \subset K$, and every $N \in \mathbb{N}$ there is $n \ge N$ and a a dynamical Cantor set $(K', U, \psi', \mathcal{I})$ with the following properties:*

- *$\mathcal{V}_n X, \psi$ is n_ε-wandering;*
- *the C^1-distance between $(K, U, \psi, \mathcal{I})$ and $(K', U, \psi', \mathcal{I})$ is less than ε*
- *$\mathcal{V}_n(X, \psi) = \mathcal{V}_(X', \psi')$ where X' is the continuation of X;*
- *for every component I of $\mathcal{V}_n(X)$ and every $0 \le n < n_{varepsilon}$ one has $(\psi')^n(I) = \psi^n(I)$;*
- *for every component I of $\mathcal{V}_n(X)$ one has $a(I) \ge 1 - \varepsilon$*
- *$a(K') \ge a(K)$ and $A(K') \ge A(K)$*

Proof : We first chose N such that $\mathcal{V}_N(X)$ is n_ε wandering, and such that the linerarization K_0 of K by using Lemma 5.10 is an $\varepsilon/100$ perturbation of K. Recal that $a(K_0) \ge a(K)$. Notice that K_0 satisfies all the announced properties but one: we do not ensures that, for every component I of $\mathcal{V}_n(X)$ one has $a(I) \ge 1 - \varepsilon$.

We prooceed now exactly as in the proof of Theorem 5.1 using Lemma 5.12. □

5.4 Hausdorff dimension and intersection

5.4.1 Definition

I will not give a precise, formal, definition of the Hausdorff dimension. Let me just try to give an idea. Given some $\alpha > 0$, for any finite covering $\mathcal{U} = \{U_i\}$ of K we associated the sum $H_\alpha(\mathcal{U}) = \sum_1^n \delta(U_i)^\alpha$ where δ denotes the diameter of U_i. One denotes $\delta(\mathcal{U})$ the max of the $\delta(U_i)$.

Then one denotes $H_{\alpha,\delta}(K) = \inf\{H_\alpha(\mathcal{U}), \delta(\mathcal{U}) < \delta\}$. This number is clearly decreasing with δ: if δ is smaller, the infinimum is considered on a smaller set, so it is larger. Then the limit is well defined and one denotes by $H_\alpha(K)$ the limit. The number $H_\alpha(K)$ is clearly deacrising with α. In fact one verifies easily that

$$\lim_{\delta\to 0} \frac{H_{\alpha,\delta}(K)}{H_{\beta,\delta}(K)} = 0, \quad \forall 0 < \beta < \alpha$$

One deduce the existence of a unique number $H(K)$ such that $H_\alpha(K) = +\infty$ for $\alpha < H_\alpha(K)$ and $H_\alpha(K) = 0$ for $\alpha > H_\alpha(K)$. This number $H(K)$ is the Hausdorff dimension of K.

Example 2. *Consider the affine dynamical Cantor set on $[0,1]$ defined by two affine maps from $[0,\alpha] \to [0,1]$ and $[1-\alpha,1] \to [0,1]$. There are 2^n intervals of generation n, all of them of diameter α^n. For this specific cover on has $H_t(\mathcal{U}_n) = 2^n.\alpha^{nt} = (2\alpha^t)^n$. The unique choice for this limit beeing different from 0 and ∞ is $2\alpha^t = 1$ that is*

$$t = \frac{\log 2}{-\log \alpha}$$

5.4.2 Disjoining two Cantor sets with low Hausdorff dimension

Here are some easy properties:

- If ϕ is a diffeomorphism, then $H(K) = H(\phi(K)$
- if ϕ is a C^1-map then $H(\phi(K) \leq H(K)$
- $H(K \times L) = H(K) + H(L)$

Corollaire 5.16.
- *If $K, L \subset \mathbb{R}$ are compact sets with $H(K) + H(L) < 1$, then the set of $t \in \mathbb{R}$ such that $(K+t) \cap L = \emptyset$ is open and dense.*
- *More generally, ff $K_0, K_1 \ldots, K_\ell \subset \mathbb{R}$ are compact sets with $\sum_0^\ell H(K_i) < \ell$, then the set of $t_1, \ldots, t_\ell \in \mathbb{R}$ such that*

$$K_0 \cap \left(\bigcap_{i=1}^{\ell} (K_i + t_i) \right) = \emptyset$$

is open and dense.

Proof : This set is open because K and L are compact. The complement is the set of t such that $(K+t) \cap L \neq \emptyset$ that is $t \in L - K$. $L - K$ is the projection of $K \times L \subset \mathbb{R}^2$ by $(x,y) \mapsto x - y$. Hence $H(L-K) \leq H(K) + H(L) < 1$. In particular, $L - K$ has empty interior, proving the density of its complement.

More generally, $K_0 \cap (K_1 + t_1) \cap \cdots \cap (K_\ell + t_\ell) \neq \emptyset$ means that there is $(x_0, x_1, \ldots, x_\ell) \in \prod_0^\ell K_i$ such that $t_i = x_0 - x_i$. In other words $(t_1, \ldots, t_\ell)$ belongs to the projection of $K_0 \times \cdots \times K_\ell \subset \mathbb{R}^{\ell+1} \to \mathbb{R}^\ell$ by $(x_0, x_1 \ldots, x_\ell) \mapsto (x_1 - x_0, \ldots, x_0 - x_\ell)$. This projection has a dimension strictly less that ℓ, by hypothesis so that the projection has empty interior. □

This lemma shows that the theorem is easy if $H(K) + H(L) < 1$. Indeed, in that case, one notices that $K_t = K + t$ is the dynamical Cantor set associted to the map $h_t \psi h_t^{-1}$ obtained by conjugating ψ by the translation h_t. Furthermore, for small t, K_t is a small perturbation of K. Now, there is a dense open subset of t for which $K_t \cap L = \emptyset$.

Hence the difficulty of the theorem starts with dynamical Cantor set K L such that $H(K) + H(L) > 1$.

5.5 Disjoining Cantor sets K, L such that L has low Hausdorff dimension: $H(L) < \frac{1}{2}$

In this section, we present the easier case, when one of the two Cantor sets, say L, has a Hausdorff dimension samller than $\frac{1}{2}$; the other one may have a Hausdorff dimension arbitrarily close to 1 so that translations may not disjoint K and L. I am not telling that this case is easy, because even this case presents many difficulties.

For simplifying one argument at the end of the proof, I will start assuming that L is a very specific Cantor set:

Let L_α be the Cantor set on $[0,1]$ defined on $[0,1]$ by the map $\psi\colon [0,\alpha]\cup[1-\alpha,1]\to[0,1]$ defined by $t\in[0,\alpha]\mapsto \frac{t}{\alpha}$ and $1-t\in[1-\alpha,1]\mapsto\frac{t}{\alpha}$. Then

$$H(L_\alpha) < \frac{1}{2} \Longleftrightarrow \alpha < \frac{1}{4}.$$

The Cantor set L has regular intervals and gaps:

Remark 5.17. *For every $i\geq 0$, each point $x\in L$ belongs to an interval of the construction I of size α^i, such that the adjacent gaps have each length larger than $2\alpha^i$.*

Let (K,φ) be a dynamical Cantor set. We want to push K in the gaps of L. However, if $H(K)$ is close to 1, so that $H(K)+H(L)>1$ that is not possible without changing the geometry of K: the tranlated Cantor sets $K+t$, and more generally every Cantor set K' in a C^2 neighborhood of K, will meet L. We will enlarge gaps of K, so that the remaining intervals of the construction will be small and we will push this intervals in the gaps of L. However one cannot enlarge all the gaps of K. We will choose a set of gaps that we will enlarge.

For that there is a simple idea: K varies continuously with φ and $K\cap L$ varies upper semi continuously with φ. We are just interested in enlarging gaps for a set of intervals of the construction which cover $K\cap L$. However, as explained in Proposition 5.15 one may enlarge a set of gaps if they are contained in a set of intervals whose iterates remains pairwise disjoint from a long time. For this reason, we are interested in separating $K\cap L$ from all its positive iterates.

5.5.1 separating the iterates of $K\cap L$

We will prove:

Lemme 5.18. *For any C^2-generic dynamical Cantor set (K,φ) one has that $K\cap L$ has all its iterates $\varphi^i(K\cap L)$ pairwise disjoint:*

$$\forall i,j\in\mathbb{N}, i\neq j \Longrightarrow \varphi^i(K\cap L)\cap\varphi^j(K\cap L)=\emptyset.$$

Let me just present here the proof of a slightly simpler lemma (but not sufficient). The Proof of Lemma 5.18 is done in the general case see Lemma 5.24

Lemme 5.19. *Given any C^2- dynamical Cantor set (K_0,φ) and $n\in\mathbb{N}$, there is K arbitrarilly C^2-close to K_0 such that*

$$\forall 1\leq i\leq n,\quad \varphi^i(K\cap L)\cap K\cap L=\emptyset.$$

We will prove this lemma by induction on n. Let us first show the case $n=1$: one just want to separate $\psi(K\cap L)$ from $K\cap L$.

$K\cap L$ and hence $\psi(K\cap L)$ have a Hausdordrf dimension smaller than $\frac{1}{2}$ because $H(L)<\frac{1}{2}$. Hence small translation of $\psi(K\cap L)$ are disjoint from $K\cap L$. It seems enough to change ψ by some $\psi+t$. Indeed this argument works, but it is not so easy, because changing ψ by $\psi+t$ changes K hence changes $K\cap L$, so that $(\psi+t)(K\cap L)$ is not the tranlation by t of $\psi(K\cap L)$. Let us perform now rigourously the argument:

Proof : Let U be a compact neighborhood of K on which ψ is defined. Now $L \cap U$ and $\psi(L \cap U)$ are Cantor set whose Hausdorff dimension is less that $\frac{1}{2}$. As a consequence, for a open and dense value of t, $(\psi + t)(L \cap U)$ is disjoint from L. Hence $(\psi + t)(K_t \cap L)$ is disjoint from $K_t \cap L$ where K_t is defined by $\psi + t$.

One assume now that $\forall 1 \leq i \leq n-1, \varphi^i(K \cap L) \cap K \cap L = \emptyset$, and we want to show that $\varphi^n(K \cap L)$. The difficulty here is that, replacing ψ by $\psi + t$ does not turn ψ^n into $\psi^n + t$.

By the induction hypothesis, $\varphi^i(K \cap L)$, $0 \leq i \leq n-1$ are disjoint compact sets. We chose a small compact neighborhood U_0 of $K \cap L$ such that ψ^i is define on U_0 and the $U_i = \psi^i(U_0)$ are dijoint for $0 \leq i \leq n-1$.

For every small t there is a C^2-small perturbation ψ_t of ψ such that

- $\psi_t = \psi$ on U_i for $0 \leq i \leq n-2$
- but $\psi_t = \psi + t$ on U_{n-1}.

Then $(\psi_t)^n = \psi^n + t$ on U_0. Hence, for an open and dense value of small t one gets that $\psi_t^n(L \cap U_0) \cap L = \emptyset$.

However, $K \cap L$ varies uper semi continuously. So for every small t on has

$$(K_t \cap L) \subset U_0.$$

One deduces $\psi_t^n(K_t \cap L) \cap L = \emptyset$ ending the proof. □

5.5.2 Opening gaps

Then by using Proposition 5.15 and lemma 5.18, for any $\varepsilon > 0$ we get some n_ε allowing to opening large gaps in n_ε-wandering collections of intervals.

Then there is N such that the union $\mathbb{O}_N(K)$ of all the interval of the construction of generation N which meet $K \cap L$ is n_ε wandering.

Notice that $K \cap \mathbb{O}_N$ is a closed open set of K. Hence $K \setminus \mathbb{O}_N$ is a compact set disjoint from L. As a consequence there is $N_0 > N$ such that every interval of the construction of K of generation N_0 which is not contained in $\mathbb{O}_N$ is disjoint from L.

Notice that this property is robust under C^0-perturbation of K, because the map $K \mapsto K \cap L$ is upper-semicountinuous under C^0 perturbations of K. As a consequence one gets:

Lemme 5.20. *For every $n \in \mathbb{N}$ and any dynamical Cantor set $\tilde{K}$ close to K, let denote by $\mathcal{U}_n(\tilde{K})$ the set of the n-generation intervals of the construction of $\tilde{K}$ contained in $\mathbb{O}_N(\tilde{K})$. Then there is N_1 such that, for every $n \geq N_1$ If I is a the n-generation intervals of the construction of $\tilde{K}$ and $I \cap L \neq \emptyset$ then $I \in \mathcal{U}_n(\tilde{K})$.*

Now we can perform the ε-perturbation given by Proposition 5.15 opening the prescribe gap u_I for every interval of $\mathcal{U}_n(K)$ and getting:

Lemme 5.21. *Le (K, φ) be C^2 dynamical Cantor set. Then given any $\varepsilon > 0$, there is $(\tilde{K}, \tilde{\varphi})$ arbitrarily C^1-close to K and n, such that $\tilde{K} \cap L$ is covered by disjoint n-generation intervals I_i of the construction of $\tilde{K}$ with the following properties:*

- *the gaps adjacent I_i have length larger that $A(K)\ell(I_i)$*
- *I_i contains a gap u_i of length larger that $(1-\varepsilon)\ell(I_i)$.*
- *$\ell(I_i) < \varepsilon$*

5.5.3 Separating K from L

Let us end the proof of the Theorem in that case. Every point in $\tilde{K} \cap L$ belongs to an intervals I_i of the construction of $\tilde{K}$ and to an interval of the construction J of L with $\ell(J)/\ell(I_i) \in [\varepsilon, \alpha^{-1}\varepsilon]$. Let I_i^+ and I_i^- be the two connected components of $I_i \setminus u'_{I_i}$ (that is after removing the large gap of I_i given by Lemma 5.21). This components have a length smaller than $\varepsilon\ell(I_i)$.

A small conjugacy of ψ with support on the union of I_i and of its adjacent gaps will change K' such that the continuations of I_i^+ and I_i^- will now be in the gaps of J. This ends the proofs in that case.

5.5.4 Generalization to any dynamical Cantor set L with $H(L) < \frac{1}{2}$.

We proved the theorem using very specific Cantor sets L_α. However, in Section 5.5.1 we just used the fact that $H(L) < \frac{1}{2}$, and the following section just used the fact that the iterates of $K \cap L$ are disjoint. Hence we just used the specificity of L in the last section for having the following property:

There are constant $A(L) > 0$ and $\alpha \in (0,1)$ such that, for every $\varepsilon > 0$ small enough, any point of L belongs to an interval J of the construction whose length $\ell(J)$ belongs to $[\varepsilon, \alpha^{-1}\varepsilon]$ and such that the adjacents gaps are larger than $\ell(J).A(L)$.

One easily verifies that every locally affine, and more generally every C^2-dynamical Cantor set satisfies these properties. Hence we proved the theorem for any C^2-Cantor set L with $H(L) < \frac{1}{2}$.

5.6 the general case

The general case follows the same spirit. Up to performing a small perturbation, one may assume that L is locally affine.

5.6.1 Empty intersection of a large number $n \geq k$ of iterates $\psi^{i_j}(K \cap L)$, $0 \leq j \leq n$

If $H(L) > \frac{1}{2}$, one cannot disjoint $K \cap L$ from $\psi(K \cap L)$ just by changing ψ by $\psi + t$, for some small t, because $H(K \cap L) + H(\psi(K \cap L))$ can be larger than 1.

Let k such that $(k+1)H(L) < k$. Then according to Lemma 5.16 small translations can make empty the intersection of n iterates $\psi^i(K \cap L)$. However we need to perform a dynamical perturbation, and it is not possible to perform independent perturbations of different iterates of $K \cap L$ if these iterate are not disjoint. For this reason Moreira states in fact a stronger result.

For any $i \in \mathbb{N}$, one consider the set of the intervals of the construction where ψ^i is defined and injective. Recall that two intervals of the construction are either disjoint or one is contained in the other. Hence there is a well defined notion of pair (I, i) where I *maximal interval of the construction where* ψ^i *is define and injective*. We denote by $\mathcal{P}$ the set of this pairs (I, i).

Proposition 5.22. *For a residual set* $\mathcal{R}$ *of* C^2*-dynamical Cantor sets* K*, given any* $k+1$ *different elements* $(I_0, i_0), \ldots, (I_k, i_k) \in \mathcal{P}$*, then*

$$\bigcap_{j=0}^{k} \psi^{i_j}(I_j \cap K \cap L) = \emptyset$$

As, for any i, the Cantor set K can be covered by interval I with $(I, i) \in \mathcal{P}$, the Proposition implies directly

Corollaire 5.23. *For any* $K \in \mathcal{R}$ *and for any* $i_0 < i_1 < \cdots < i_k$ *one has:*

$$\bigcap_{j=0}^{k} \psi^{i_j}(K \cap L) = \emptyset.$$

5.6.2 Proof of the Proposition 5.22

Notice that, if K' is a perturbation of K then every interval I of the constructuon of K has a continuation I' for K'. Furthermore, if $(I, i) \in \mathcal{P}$ the $(I', i) \in \mathcal{P}'$.

Hence one may first chose the finite sequence $(I_0, i_0), \ldots, (I_k, i_k) \in \mathcal{P}$ and Proposition 5.22 is a direct consequence of

Lemme 5.24. *For any C^2 -dynamical Cantor set K there is K' arbitrarily C^2-close to K such that*

$$\bigcap_{j=0}^{k} (\psi')^{i_j}(I'_j \cap K' \cap L) = \emptyset.$$

So K is an arbitrary C^2-dynamical Cantor set. First remark that,

Remark 5.25. *for C^r-generic K, $K \cap L$ does not contain any pre-periodic point of ψ*

Proof : That is just because the set of preperiodic point is countable. Hence small translation put each of them in the gaps of L, and for each of them, being out L is robust. □

Hence one may assume that $K \cap L$ does not contain any pre-periodic point of ψ.

Lemme 5.26. *Let $x \in \bigcap_{j=0}^{k} (\psi)^{i_j}(I_j \cap K \cap L)$, and for every j let $y_j \in I_j \cap K \cap L$ such that $\psi^{i_j}(y_j) = x$. Then, for $j \neq h$ one has $y_j \neq y_h$.*

Proof : If $y = y_j = y_h$ then $I_j \cap I_h \neq \emptyset$. As $(I_j, i_j) \neq (I_h, i_h)$ this implies that $i_j \neq i_h$, for instance $i_j < i_h$. Hence $x = \psi^{i_j}(y) = \psi^{i_h}(y)$ one gets $x = \psi^{i_h - i_j}(x)$, so $y \in K \cap L$ is preperiodic contradicting the fact that $K \cap L$ does not contain any preperiodic point. □

So for every $x \in \bigcap_{j=0}^{k} (\psi)^{i_j}(I_j \cap K \cap L)$, there are intervals of the construction $J_j(x)$, $0 \leq j \leq k$, such that

(J.1) $J_j(x) \subset I_j$

(J.2) $y_j \in J_j$

(J.3) for $j \neq h$, $J_j(x) \cap J_h(x) = \emptyset$.

(J.4) for every j, the iterates $\psi^t(J_j(x))$, $t \in \{0, \ldots, i_j\}$ are pairwize disjoint.

(J.5) if $j, h \in \{0, \ldots, k\}$ admits some $t \in \{0, \ldots, r_j\}$ with $\psi^t(I_j) \cap I_h \neq \emptyset$ then $r_j > r_h$. Furthermore, t is unique.

These properties will help us to perform perturbations of ψ supported on the J_j, with a control of the interaction between the perturbation. The intervals $J_j(x)$ are neigborhood of y_j in K. So $\psi^{i_j}(J_j(x))$ is a neighborhood of x in K. One deduces

Lemme 5.27. *There is $\varepsilon(x)$ and a C^1-neighborhood $\mathcal{V}(x)$ of K such that for every $(K', \psi') \in \mathcal{V}(x)$ one has*

$$K' \cap [x - \varepsilon(x), x + \varepsilon(x)] \subset (\psi')^{i_j}(J'_j(x))$$

where $J'_j(x)$ is the continuation of $J_j(x)$ for K'.

As a direct consequence one gets

Corollaire 5.28. *For every $(K', \psi') \in \mathcal{V}(x)$ one has*

$$\left(\bigcap_{j=0}^{k} (\psi')^{i_j}(I'_j \cap K' \cap L) \right) \cap [x - \varepsilon(x), x + \varepsilon(x)] \subset \bigcap_{j=0}^{k} (\psi')^{i_j}(J'_j(x) \cap K' \cap L).$$

We fix a finite covering of $\bigcap_{j=0}^{k}(\psi)^{i_j}(I_j \cap K \cap L)$ by intervals $[x-\varepsilon(x), x+\varepsilon(x)]$, $x \in X$ where X is a finite subset of $\bigcap_{j=0}^{k}(\psi)^{i_j}(I_j \cap K \cap L)$. One denotes $\widetilde{\mathcal{V}} = \bigcap_{x\in X} \mathcal{V}(x)$; it is a C^1-neighborhood of K.

Remark 5.29. *The map $K' \mapsto \bigcap_{j=0}^{k}(\psi')^{i_j}(I'_j \cap K' \cap L)$ is upper-semi continuous.*

As a consequence of this remark, there is a C^1-neighborhood $\mathcal{V} \subset \widetilde{\mathcal{V}}$ of K such that, for every $K' \in \mathcal{V}$ one has

$$\bigcap_{j=0}^{k}(\psi')^{i_j}(I'_j \cap K' \cap L) \subset \bigcup_{x\in X}[x-\varepsilon(x), x+\varepsilon(x)].$$

Hence

$$\bigcap_{j=0}^{k}(\psi')^{i_j}(I'_j \cap K' \cap L) \subset \bigcup_{x\in X}\left(\bigcap_{j=0}^{k}(\psi')^{i_j}(J'_j(x) \cap K' \cap L)\right).$$

Now Lemma 5.24 follows directly from the following lemma:

Lemme 5.30. *For every $x \in X$ there is a C^1-open and C^2-dense subset $\mathcal{V}_x$ of $\mathcal{V}$ such that for every $K' \in \mathcal{V}_x$ one has*

$$\bigcap_{j=0}^{k}(\psi')^{i_j}(J'_j(x) \cap K' \cap L) = \emptyset.$$

This property is clearly C^1-open. It remains to prove the C^2-density. In ordre to simplify the notation, we now denote by K an arbitrar dynamical Cantor sets in $\mathcal{V}$.

Just because the properties (J.3) (J.4) and (J.5) of the intervals $J_j(x)$, one verifies:

Lemme 5.31. *For any $t = (t_0, t_1, \ldots t_k)$ small enough, there is a C^2-small perturbation ψ_t of ψ such that, for every $j \in \{0, \ldots, , k\}$, the restriction of $(\psi_t)^{i_j}$ to $J_{t,j}(x)$ is $\psi^{i_j} + t_j$.*

Proof : We proceed by induction. Up to re-indexing the J_j we can assume that the times i_j are increasing. Then one defines:

- $\psi_0 = \psi_{(t_0,0,\ldots,0)}$ by $\psi_0 = \psi$ out a small neighborhood of J_0 and $\psi_0 = \psi^{1-i_0} \circ (\psi + t_0) \circ \psi^{i_0-1}$ on a smaller neighborhood of J_0
- $\psi_j = \psi_{(t_0,t_1,\ldots,t_j,0\ldots,0)}$ by $\psi_j = \psi_{j-1}$ out a small neighborhood of J_j and $\psi_j = \psi_{j-1}^{1-i_j} \circ (\psi_{j-1} + t_j) \circ \psi_{j-1}^{i_0-1}$ on a smaller neighborhood of J_0.

□

One concludes by recalling Corollary 5.16: for an open and dense subset of t the intersection $\bigcap_{j=0}^{k}(\psi^{i_j}(L)+ t_j)$ is empty.

5.6.3 Generic properties with control of the regularity constants $a(K)$ and $A(K)$

One difficulty is that every C^1-generic Cantor set satisfies $a(K) = A(K) = 0$, by Ures argument. Hence we cannot apply Proposition 5.15 for C^1-generic Cantor sets. For solving this properties we will require that $a(K)$ and $A(K)$ remains far from 0.

Remark 5.32. *The maps $K \mapsto a(K)$ and $K \mapsto A(K)$ are upper semi continuous when K vaires in the C^1-topology (even in the C^0 one) becau they are defined as infimum of continuous functions.*

As a consequence of the remark above the set

$$\mathcal{K}_a = \{C^1\text{-dynamical Cantor set with } a(K) \geq a \text{ and } A(K) \geq a\}$$

is a close subset of the set of C^1-dynamical Cantor sets. In particular

Corollaire 5.33. *Given any $a > 0$, the set $\mathcal{K}_a$ endowed with the C^1-topology on the dynamical Cantor sets it is a Baire space.*

As a consequence of Proposition 5.22 and Corollary 5.23 one gets

Corollaire 5.34. *There is a residual set $\mathcal{R}_{a,k}$ of $\mathcal{K}_a$, such that, for any $K \in \mathcal{R}_{a,k}$ and for any $i_0 < i_1 < \cdots < i_k$ one has:*

$$\bigcap_{j=0}^{k} \psi^{i_j}(K \cap L) = \emptyset.$$

Proof : The set with this property is clearly a G_δ. It remains to see that it is dense. The idea is the following: given $K \in \mathcal{K}_a$ one may approach K in the C^1 topology by K_1 which is locally affine (in particular is C^2) and such that $a(K_1) \geq a(K)$ and $A(K_1) \geq A(K)$. Then one can approach K_1 by K_2 locally affine, with $a(K_2) > a(K_1)$ and $A(K_2) > A(K)$. Then Corollary 5.23 asserts that K_2 is C^2 approached by K_3 with the announced property. As the functions a and A vary continously for the C^2 topology, one gets that K_3 belongs to $\mathcal{K}_a$, ending the proof. □

5.6.4 Decreasing the number of iterates $\psi^{i_j}(K \cap L)$, $0 \leq j \leq n$ needed for having an empty intersection.

We ends the proof of the theorem by the following proposition

Proposition 5.35. *Given $m \geq 1$, assume that for every $a > 0$ there is a residual set $\mathcal{R}_{a,m}$ of $\mathcal{K}_a$, such that, for any $K \in \mathcal{R}_{a,m}$ and for any $i_0 < i_1 < \cdots < i_m$ one has:*

$$\bigcap_{j=0}^{m} \psi^{i_j}(K \cap L) = \emptyset.$$

Then, for every $a > 0$ there is a residual set $\mathcal{R}_{a,m-1}$ of $\mathcal{K}_a$, such that, for any $K \in \mathcal{R}_{a,m-1}$ and for any $i_0 < i_1 < \cdots < i_{m-1}$ one has:

$$\bigcap_{j=0}^{m-1} \psi^{i_j}(K \cap L) = \emptyset.$$

Theorem 5.3 is the corresponding statement for $m = 0$, hence is a consequence of a straightforward induction argument using Corollary 5.34 (for starting the induction) and Proposition 5.35 (for the induction steps).

5.6.5 Proof of Proposition 5.35

Given $m \geq 1$, assume that there is a residual set $\mathcal{R}_{a,m}$ of $\mathcal{K}_a$, such that, for any $K \in \mathcal{R}_{a,m}$ and for any $i_0 < i_1 < \cdots < i_m$ one has:

$$\bigcap_{j=0}^{m} \psi^{i_j}(K \cap L) = \emptyset.$$

Fix now $0 \leq i_0 < i_1 < \cdots < i_{m-1}$. We need to proove

Lemme 5.36. *There is a C^1 open subset $\mathcal{O}$ of $\mathcal{K}_a$, such that, for any $K \in \mathcal{O}$ one has:*

$$\bigcap_{j=0}^{m-1} \psi^{i_j}(K \cap L) = \emptyset.$$

The openess is trivial, we just have to proof the density, which is the aim of the rest of this notes.

We start with $K_0 \in \mathcal{K}_a$. As already seen, one may C^1-approache K_0 by $K_1 \in \mathcal{K}_{a_0}$ with $a_0 > a$. Now one may approach K_1 by $K \in \mathcal{R}_{a_0,m}$. As a consequence, for any $r < s \in \mathbb{N}$ one has

$$\psi^r(\bigcap_{j=0}^{m-1} \psi^{i_j}(K \cap L)) \cap \psi^s(\bigcap_{j=0}^{m-1} \psi^{i_j}(K \cap L)) = \emptyset$$

, because this intersection maybe written as an intersection of $m+1$ iterates of $K \cap L$.

So all the iterates of $\kappa_L(K) =_{\text{def}} \bigcap_{j=0}^{m-1} \psi^{i_j}(K \cap L)$ are pairwise disjoint. Hence it satisfies the hypothesis of Proposition 5.15.

5.6.6 Choosing the intervals where we will enlarge gaps

Then by using Proposition 5.15 , for any $\varepsilon > 0$ we get some n_ε allowing to opening large gaps in n_ε-wandering collections of intervals, and the choice of $K \in \mathcal{K}_{a_0,m}$ provides us such intervals:

Then there is N such that the union $\mathbb{O}_N(K)$ of all the interval of the construction of generation N which meets $\kappa_L(K)$ is n_ε wandering. For $\tilde{K}$ close to K we denote by $\mathbb{O}_N(\tilde{K})$ the union of the continuations of this intervals.

Notice that $K \cap \mathbb{O}_N$ is a closed open set of K. Hence $K \setminus \mathbb{O}_N$ is a compact set disjoint from L. As a consequence every interval of the construction of K of generation N which is not contained in $\mathbb{O}_N$ is a compact set disjoint from $\kappa_L(K)$.

Notice that this property is robust under C^0-perturbation of K, because

Lemme 5.37. *The map $K \mapsto K \cap L$ is upper-semicountinuous under C^0 perturbations of K.*

Proof : The map $K \mapsto K \cap L$ vary semicontinuously with K that is with ψ in the C^0 topology. So the maps $K \mapsto \psi^j(K \cap L)$ are all semi-continuous. Once more the intersection of Compact sets is upper semi-continuous with repsect to the Hausdorff topology, and the composition of upper-semi continuous fonctions is upper semi continuous. □

As a consequence one gets:

Lemme 5.38. *For every $n \in \mathbb{N}$ and any dynamical Cantor set $\tilde{K}$ close to K, let denote by $\mathcal{U}_n(\tilde{K})$ the set of the n-generation intervals of the construction of $\tilde{K}$ contained in $\mathbb{O}_N(\tilde{K})$. Then there is N_1 such that, for every $n \geq N_1$ If I is a the n-generation intervals of the construction of $\tilde{K}$ and $I \cap \kappa_L(K) \neq \emptyset$ then $I \in \mathcal{U}_n(\tilde{K})$.*

5.6.7 Opening gaps close to $\kappa_L(K)$

Now we can perform the ε-perturbation given by Proposition 5.15 opening the prescribe gap u_I for every interval of $\mathcal{U}_n(K)$ and getting:

Lemme 5.39. *Le (K, φ) be C^2 dynamical Cantor set. Then given any $\varepsilon > 0$, there is $(\tilde{K}, \tilde{\varphi})$ arbitrarily C^1-close to K and n, such that $\kappa_L(\tilde{K})$ is covered by disjoint n-generation intervals I_i of the construction of $\tilde{K}$ with the following properties:*

- *the gaps adjacent I_i have length larger that $A(K)\ell(I_i)$*
- *I_i contains a gap u_i of length larger that $(1-\varepsilon)\ell(I_i)$.*
- *$\ell(I_i) < \varepsilon$*

5.6.8 separating one iterates from the intersection of the others

Now, for every interval in $\mathcal{U}_n(K)$ we will change the inverse branches of ψ^{i_1} of this interval $I \in \mathcal{U}_n(K)$ in order to that $L \cap \tilde{\psi}^{-1}(I \setminus U_I) \cap L = \emptyset$.

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