Lectures on Smooth Ergodic Theory

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1 Elementary Facts from Ergodic Theory

In this section, we will introduce some basic concepts and results in ergodic theory that will be frequently cited in latter sections. Most of the results are stated without proof, since all of them can be found in any standard textbook on ergodic theory.

1.1 Invariant Measure

Let X be a compact metric space, $\mathscr{B}(X)$ be the Borel σ-algebra of X, and $C(X)$ be the space of continuous functions on X with the usual maximal-module norm. $P(X)$ denotes the space of Borel probability measures on X with weak-$*$ topology, i.e. :

$$\lim_{n\to\infty}\mu_n = \mu \text{ in } P(X) \iff \lim_{n\to\infty}\int_X \varphi\,\mathrm{d}\mu_n = \int_X \varphi\,\mathrm{d}\mu\ ,\ \forall\varphi\in C(X)\,.$$

Proposition 1.1. *$P(X)$ is a compact metric space.*

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a dense subset of $C(X)$, and define:

$$d(\mu,\nu) = \sum_{n=1}^{\infty}\frac{1}{2^n\|\varphi_n\|}\left|\int_X f_n\,\mathrm{d}\mu - \int_X f_n\,\mathrm{d}\nu\right|\ ,\ \forall\mu,\nu\in P(X)\,.$$

It is easy to verify that d is a metric on $P(X)$ comparable to its weak-$*$ topology. □

For any given continuous map $f : X \to Y$, where both X and Y are compact metric spaces, there is an induced map $f_* : P(X) \to P(Y)$, $\mu \mapsto f_*\mu$, defined as:

$$f_*\mu(E) = \mu(f^{-1}(E)),\ \forall E\in\mathscr{B}(Y)\,.$$

It is clear that f_* is continuous by definition.

Definition 1.1 (Invariant Measure)**.** In the case that $f : X \circlearrowleft$, we say $\mu \in P(X)$ is f-invariant, or f preserves μ, if $f_*\mu = \mu$.

We denote the set $\left\{\mu\in P(X)\,\middle|\, f_*\mu = \mu\right\}$ by $P(X, f)$.

Remark. Generally speaking, to define measure-preserving map, we only need that $(X, \mathscr{B})$ is a measurable space and $f : X \circlearrowleft$ is measurable. But in this lecture we are only interested in the above case.

Proposition 1.2. *If X is a compact space and $f : X \circlearrowleft$ is continuous, then $P(X, f) \neq \emptyset$.*

Proof. Take an arbitrary $\mu \in P(X)$, and define $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} f_*^i\mu$. Then $\{\mu_n\}_{n\in\mathbb{N}} \subset P(X)$. By compactness of $P(X)$, there exists a subsequence $\{\mu_{n_i}\}$ converging to some $\nu \in P(X)$. We only need to show that $f_*\nu = \nu$. Notice that

$$\int_X \varphi\,\mathrm{d}\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\int_X \varphi\circ f^i\,\mathrm{d}\mu\ ,\ \forall\varphi\in C(X)\,.$$

So we have:

$$\left|\int_X \varphi\,\mathrm{d}f_*\mu_n - \int_X \varphi\,\mathrm{d}\mu_n\right| = \frac{1}{n}\left|\int_X (\varphi\circ f^n - \varphi)\,\mathrm{d}\mu\right| \leq \frac{2}{n}\|\varphi\|\,.$$

Let $n \to \infty$, we get:

$$\int_X \varphi \, \mathrm{d} f_* \nu = \lim_{n\to\infty} \int_X \varphi \, \mathrm{d} f_* \mu_n = \lim_{n\to\infty} \int_X \varphi \, \mathrm{d}\mu_n = \int_X \varphi \, \mathrm{d}\nu \,.$$

Since this is true for any φ, ν is f-invariant. □

1.2 Natural Extension

Let (X, d) be a compact metric space. So is the product space $X^{\mathbb{Z}} = \left\{ (x_i)_{i\in\mathbb{Z}} \,\middle|\, x_i \in X \right\}$, whose metric, still denoted by d, can be given as:

$$d((x_i), (y_i)) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} d(x_i, y_i) \,.$$

The left shift $\sigma : X^{\mathbb{Z}} \circlearrowleft$, $(x_i) \mapsto (x_{i+1})$ is a homeomorphism preserving the product measure. There is a natural projection $\pi : X^{\mathbb{Z}} \to X$, $(x_i) \mapsto x_0$, which is continuous, surjective and measure preserving.

Definition 1.2 (Natural Extension)**.** When $f : X \circlearrowleft$ is continuous and surjective, we can define a subspace X_f of $X^{\mathbb{Z}}$, called the natural extension of (X, f), as:

$$X_f = \left\{ (x_i) \in X^{\mathbb{Z}} \,\middle|\, x_{i+1} = f(x_i) \right\}.$$

The following proposition holds automatically by definition.

Proposition 1.3. *X_f is a compact metric space, $\sigma_f : X_f \circlearrowleft$ is a homeomorphism, and the following diagram commutes:*

$$\begin{array}{ccc} X_f & \xrightarrow{\sigma_f} & X_f \\ \pi\downarrow & & \pi\downarrow \\ X & \xrightarrow{f} & X \end{array}$$

Here σ_f is the restriction of σ to X_f. □

Theorem 1.4. *Given $\mu \in P(X, f)$, there exists a unique $\mu_f \in P(X_f, \sigma_f)$, such that $\pi_*(\mu_f) = \mu$.*

Proof. Let $\mathscr{A}_n = \left\{ (\pi_{-n})^{-1}(B) \,\middle|\, B \in \mathscr{B}(X) \right\}$, $n \in \mathbb{N}$, where $\pi_{-n} : X_f \to X$, $(x_i)_{i\in\mathbb{Z}} \mapsto x_{-n}$ is the projection to the coordinate $-n$. Then $\{\mathscr{A}_n\}$ is a sequence of increasing sub-σ-algebras of $\mathscr{B}(X_f)$, and their union $\mathscr{A} = \bigcup_{n=0}^{\infty} \mathscr{A}_n$ is a set algebra, which generates $\mathscr{B}(X_f)$. To define μ_f, we first notice that it is uniquely determined on $\mathscr{A}$ by the properties that $(\sigma_f)_*(\mu_f) = \mu_f$, and $\pi_*(\mu_f) = \mu$. This is because: for $E \in \mathscr{A}_0$, $\mu_f(E) = \mu(\pi(E))$; for $E \in \mathscr{A}_n$, $\mu_f(E) = \mu_f({\sigma_f}^{-n}(E))$, where ${\sigma_f}^{-n}(E) \in \mathscr{A}_0$. Second, since $\mathscr{B}(X_f)$ is generated by $\mathscr{A}$ and clearly μ_f is countably additive on $\mathscr{A}$, there is a unique extension of μ_f defined on $\mathscr{B}(X_f)$. So we complete the construction of μ_f, which clearly satisfies the properties in the theorem. □

Remark. According to the construction of μ_f in the proof above, it is easy to see that:

$$\mu_f(E) = \lim_{n\to+\infty} \mu(\pi_{-n}(E)), \ \forall E \in \mathscr{B}(X_f).$$

Apparently the limit on the right hand always exists, since $\{\mu(\pi_{-n}(E))\}_{n\in\mathbb{N}}$ is a decreasing sequence.

1.3 Regular Point and Ergodicity

Theorem 1.5 (Birkhoff's Ergodic Theorem)**.** *Suppose $(X, \mathscr{B}, \mu)$ is a probability space, and $f : X \circlearrowleft$ is measure-preserving. Then for any $\varphi \in L^1(\mu)$, their exists some $\varphi^+ \in L^1(\mu)$, such that:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(x) = \varphi^+(x) \ , \ \mu\text{-a.e. } x \in X \, .$$

Clearly $\varphi^+ \circ f = \varphi^+$, μ-a.e.. Moreover, $\int_X \varphi \, \mathrm{d}\mu = \int_X \varphi^+ \, \mathrm{d}\mu$.

Corollary 1.6. *If f is invertible, then φ^- is similarly defined with f^{-1} instead of f in the above theorem, and $\varphi^+ = \varphi^-$, μ-a.e..*

Proof. Suppose the conclusion is false.Without loss of generality, we can assume that $E = \left\{ x \in X \,\middle|\, \varphi^+(x) > \varphi^-(x) \right\}$ has positive measure. Since E is an f-invariant set, we can apply Birkhoff's ergodic theorem to both $f|_E$ and $f^{-1}|_E$. So we have: $\int_E \varphi^+ \, \mathrm{d}\mu = \int_E \varphi \, \mathrm{d}\mu = \int_E \varphi^- \, \mathrm{d}\mu$, which contradicts to the definition of E. □

If X is a compact metric space and f is a homeomorphism of X, then we have the following definition.

Definition 1.3 (Regular Point)**.** $x \in X$ is called regular for f, if:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^{-i}(x)) \ , \ \forall \varphi \in C(X) \, .$$

That is to say, there exists some $\mu_x \in P(X)$, such that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^{-i}(x)} = \mu_x \, .$$

Moreover, if $\mu_x = \mu$, we say x is regular for μ.

Remark. If f is not invertible, we only reserve the forward-iteration part in the above definition.

By defintion, the following is a direct corollary of Birkhoff's ergodic theorem.

Corollary 1.7. $\forall \mu \in P(X, f)$, $\mu\left(\left\{ x \in X \,\middle|\, x \textit{ is regular for } f \right\}\right) = 1$. □

Definition 1.4 (Ergodic Measure)**.** $\mu \in P(X, f)$ is called ergodic, if $\forall E \in \mathscr{B}(X)$ with $f^{-1}(E) = E$, $\mu(E) = 0$ or 1.

We use the notation $E(X, f)$ to denote set of all the ergodic ones in $P(X, f)$.

Proposition 1.8. $E(X, f) \neq \emptyset$.

Proposition 1.9. *If $\mu \in P(X, f)$, then the following statements are equivalent:*

(1) μ is ergodic;

(2) $\forall E \in \mathscr{B}(X)$ *with* $\mu(E) > 0$, $\mu\left(\bigcup_{n=0}^{\infty} f^{-n}(E)\right) = 1$*;*

(3) $\forall E \in \mathscr{B}(X)$ *with* $f^{-1}(E) \subset E$, $\mu(E) = 0$ *or* 1*;*

(4) $\forall E \in \mathscr{B}(X)$ *with* $f^{-1}(E) \supset E$, $\mu(E) = 0$ *or* 1*;*

(5) $\forall \varphi \in L^1(\mu)$ *with* $\varphi \circ f = \varphi$, μ*-a.e.,* φ *is constant* μ*-a.e.;*

(6) $\mu\left(\left\{ x \in X \,\middle|\, x \text{ is regular for } \mu \right\}\right) = 1.$

According to Birkhoff's ergodic theorem and the proposition above, immediately we get:

Corollary 1.10. *If* $\mu \in E(X, f)$*, then* $\varphi^+ = \int_X \varphi \, \mathrm{d}\mu$, μ*-a.e.,* $\forall \varphi \in L^1(\mu)$. □

Corollary 1.11. *In theorem 1.4, if* $\mu \in E(X, f)$*, then* $\mu_f \in E(X_f, \sigma_f)$.

Proof. Let $E \in \mathscr{B}(X_f)$ with $\mu_f(E) > 0$ be such that ${\sigma_f}^{-1}(E) = E$. We only need to show that $\mu_f(E) = 1$. First, we notice that

$$ {\sigma_f}^{-1}(E) = E \Rightarrow \pi_{-n}(E) = \pi_0(E), \ \forall n \in \mathbb{N} \quad \text{and} \quad f^{-1}(\pi_0(E)) \supset \pi_0(E). $$

Then on the one hand, according to the remark following the proof of theorem 1.4, we have $\mu_f(E) = \mu(\pi_0(E)) > 0$; on the other hand, μ is ergodic implies that $\mu(\pi_0(E)) = 1$. □

Exercise 1.1. *We consider the following dynamical system on* $T^2 = S^1 \times S^1$*:*

$$ f : T^2 \circlearrowleft \quad (x, y) \mapsto (\, x + \varphi(x) \bmod \mathbb{Z}\,,\ y + \sin(2\pi x) \bmod \mathbb{Z}\,). $$

Here

$$ \varphi : [0, 1] \to [0, +\infty), \quad \varphi(x) = \exp\left(-\frac{1}{x^2(1-x)^2}\right), \ \forall x \in (0, 1) \quad \textit{and} \quad \varphi(0) = \varphi(1) = 0. $$

Then we have:

- $\forall (x, y) \in T^2$, (x, y) *is regular;*
- $x \neq 0 \Rightarrow \mu_{(x,y)}$ *is the Lebesgue measure on* $\{0\} \times S^1$*;*
- *all the ergodic measures are of the form* $\delta_{(0,y)}$, $y \in S^1$.

The proof is just some detailed calculation, so we skip it.

Theorem 1.12 (Ergodic Decomposition)**.** *For any* $\mu \in P(X, f)$*, there exists some Borel probability measure* θ *on* $E(X, f)$*, such that:*

$$ \int_X \varphi \, \mathrm{d}\mu = \int_{E(X,f)} \left(\int_X \varphi \, \mathrm{d}\nu \right) \mathrm{d}\theta(\nu), \ \forall \varphi \in C(X). $$

Sometimes we write $\mu = \int_{E(X,f)} \nu \, \mathrm{d}\theta(\nu)$ *for short.*

Definition 1.5 (Basin of Measure)**.** For $\mu \in P(X, f)$, the set

$$ \mathcal{B}(\mu) = \left\{ x \in X \,\middle|\, x \text{ is regular for } \mu \right\} $$

is called basin of μ.

Remark. If $x \in \mathcal{B}(\mu)$ and $\lim\limits_{i\to\infty} d(f^i(x), f^i(y)) = 0$, then $y \in \mathcal{B}(\mu)$. Here ∞ means both $+\infty$ and $-\infty$, when f is invertible; only $+\infty$, otherwise.

Exercise 1.2. *Let $f : M \circlearrowleft$ be a diffeomorphism on a compact manifold M, and X be a nontrivial hyperbolic basic set, i.e. both of the stable and unstable manifolds of each periodic point are dense in X. Then the set $\left\{ x \in X \;\middle|\; x \text{ is not regular} \right\}$ is residual in X.*

Proof. Let γ_1 and γ_2 be two distinct periodic orbits in X. We can choose $\varphi \in C(X)$, such that $\varphi|_{\gamma_1} = 0$, and $\varphi|_{\gamma_2} = 1$. Since both of the stable manifold of γ_1 and of γ_2 are dense in X, the sequence of sets

$$O_n = \left\{ x \in X \;\middle|\; \exists n_1 > n, \frac{1}{n_1}\sum_{i=0}^{n_1-1} \varphi(f^i(x)) < \frac{1}{3}\ ;\ \exists n_2 > n, \frac{1}{n_2}\sum_{i=0}^{n_2-1} \varphi(f^i(x)) > \frac{2}{3} \right\}$$

are all open and dense in X. So their intersection $R = \bigcap\limits_{n=1}^{\infty} O_n$ is residual. By definition, it is clear that $\forall x \in R$, x is not regular. □

There is a more general result as follows.

Theorem 1.13. *For C^1-generic diffeomorphism $f : M \circlearrowleft$, $\left\{ x \in M \;\middle|\; x \text{ is not regular} \right\}$ is residual.*

1.4 Absolute Continuity of Measures

Definition 1.6 (Absolute Continuity and Singularity)**.** Let μ, ν be two measures on the same measurable space $(X, \mathscr{B})$. We say ν is absolutely continuous with respect to μ, denoted by $\nu << \mu$, if for any E measurable, $\mu(E) = 0 \Rightarrow \nu(E) = 0$. Moreover, μ, ν are called equivalent to each other, if both $\mu << \nu$ and $\nu << \mu$ hold simultaneously. We say they are mutually singular, denoted by $\mu \perp \nu$, if there exists some E measurable, such that $\mu(E) = 0$ and $\nu(E^c) = 0$.

Theorem 1.14 (Radon-Nikodym)**.** *If μ, ν are two measures on $(X, \mathscr{B})$ and $\nu << \mu$, then there exists a positive measurable function φ on X, such that:*

$$\nu(E) = \int_E \varphi \, \mathrm{d}\mu\ ,\ \forall E \in \mathscr{B}.$$

Usually we write $\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu} = \varphi$ or $\nu = \varphi\mu$ for short.

Theorem 1.15 (Lebesgue Decomposition)**.** *If $\mu, \nu \in P(X)$, then there exist $\alpha \in [0, 1]$ and $\nu^a, \nu^s \in P(X)$, such that $\nu^a << \mu$, $\nu^s \perp \mu$, and $\mu = \alpha\nu^a + (1 - \alpha)\nu^s$. This decomposition is uniquely determined by μ and ν.*

2 Expanding Maps on S^1

In this and the next section , we only concentrate on the map $f : S^1 \circlearrowleft$ without critical point, i.e. $|\mathrm{D}f| > 0$ on S^1, where $S^1 = \mathbb{R}/\mathbb{Z}$ is the unit cirlce. Without loss of generality, we can assume that $\mathrm{D}f > 0$ for convenience. If f is of class C^1 and $\mathrm{D}f > 1$ on S^1, we say f is expanding. It is always helpful to introduce $\tilde{f} : \mathbb{R} \circlearrowleft$, which is the natural lift of f, i.e. the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \pi\downarrow & & \pi\downarrow \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

Here $\pi : \mathbb{R} \to S^1 \; x \mapsto x \mod \mathbb{Z}$ is the natural universal covering. The trivial fact that $\mathrm{D}\tilde{f}(x) = \mathrm{D}f(\pi(x))$ is frequently used.

Remark. Sometimes it is so subtle and even annoying to distinguish what we precisely discuss about is $\mathbb{R}$ or S^1, or the exact map is f or $\tilde{f}$, that we have to neglect it. We believe that this will cause no serious confusion.

2.1 General Statement

Theorem 2.1 (Shub)**.** *Let $f : S^1 \circlearrowleft$ be a covering of degree d, $d > 1$. Then there is a continuous and increasing (i.e. orientation preserving) map $h : S^1 \circlearrowleft$ of degree 1, such that the following diagram commutes:*

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & S^1 \\ h\downarrow & & h\downarrow \\ S^1 & \xrightarrow{x \mapsto d\cdot x} & S^1 \end{array}$$

Furthermore, if f is expanding, then h is a homeomorphism.

Proof. $d > 1 \Rightarrow f$ has a fixed point x_0. Let $h(x_0) = 0$. To construct $h : S^1 \circlearrowleft$, we identify the domain with $[x_0, x_0 + 1)$ and the range with $[0, 1)$. Let

$$X_n = f^{-n}(x_0) = \left\{ x_{i_1 i_2 \cdots i_n} \,\middle|\, 0 \le i_k < d,\ k = 1, \cdots, n \right\},$$

where the permutation of $\{x_{i_1 i_2 \cdots i_n}\}$ is such that $f(x_{i_1 i_2 \cdots i_n}) = x_{i_2 \cdots i_n}$ and the map

$$h|_{X_n} : x_{i_1 i_2 \cdots i_n} \mapsto \frac{i_1}{d} + \frac{i_2}{d^2} + \cdots + \frac{i_n}{d^n}$$

is increasing. Then h is well defined on $X = \bigcup_{n=0}^{\infty} X_n$. Because h is increasing on X and $h(X)$ is dense in $[0, 1)$, we can extend h continuously to $[x_0, x_0 + 1)$ naturally, i.e.:

$$h(x) = \inf_{\substack{y \in X \\ x < y < x_0 + 1}} h(y) = \sup_{\substack{y \in X \\ x_0 < y < x}} h(y),\ \forall x \in (x_0, x_0 + 1).$$

It is evident that the properties of h are satisfied automatically. If f is expanding, then X is dense in $[x_0, x_0 + 1)$. Therefore h is injective, i.e. h is a homeomorphism. □

Definition 2.1 (Smale's Solenoid)**.** Let $D^2 = \left\{ z \in \mathbb{C} \,\middle|\, |z| \le 1 \right\}$ and $f : S^1 \circlearrowleft$ is expanding. We consider a map

$$F : S^1 \times D^2 \circlearrowleft \ (t, z) \mapsto (f(t), cz + \frac{e^{2\pi i t}}{2}),$$

where $c << \inf\left\{|t-s| \,\middle|\, f(t) = f(s) \text{ and } t \neq s\right\}$, to ensure that F is a homeomorphism from $S^1 \times D^2$ to its image. The F-invariant set $\Lambda = \bigcap_{n=1}^{\infty} F^n(S^1 \times D^2)$ is called Smale's Solenoid.

The following proposition is easy to prove.

Proposition 2.2. *The dynamical system $F : \Lambda \circlearrowleft$ topologically conjugates to the left shift of the natural extension of $f : S^1 \circlearrowleft$, denoted by $\sigma_f : \Pi \circlearrowleft$. The conjugation map is defined as:*

$$h : \Pi \to \Lambda \quad (x_n)_{n\in\mathbb{Z}} \mapsto \text{the single point in } \bigcap_{n=1}^{\infty} F^n(\{x_{-n}\} \times D^2).$$

From this standpoint, it is easy to see that each connected component of Λ is homeomorphic to S^1.

So we can think of Smale's solenoid as a geometrical realization of natural extension.

2.2 A.C.I.P. for Expanding Maps

Definition 2.2 (Absolutely Continuous Invariant Probability)**.** Let $f : M \circlearrowleft$ be a smooth dynamical system on some compact Riemannian manifold M. $\mu \in P(M, f)$ is called an absolutely continuous invariant probability, or a.c.i.p. for short, if $\mu << \text{Leb}_M$.

Theorem 2.3. *If $f : S^1 \circlearrowleft$ is an expanding map of class C^2, then there exists a unique a.c.i.p. $\nu \in P(S^1, f)$, such that it is equivalent to the Lebesgue measure on S^1 and ergodic .*

Remark. If f is only of class $C^{1+\theta}$, $0 < \theta \leq 1$, the above theorem still holds with no much modification of the proof. But the smoothness of f cannot be weaken to C^1.

Theorem 2.4 (Bochi & Fayad)**.** *If $f : S^1 \circlearrowleft$ is expanding, then C^1-generically speaking, there exists no $\mu \in P(S^1, f)$, such that $\mu << \text{Leb}$, where* Leb *denotes the Lebesgue measure on S^1.*

What we will do in the rest of this section is to prove theorem 2.3 .

Uniqueness is evident since any two distinct ergodic measures must be singular mutually. So we only need to prove existence, which will be divided into three steps.

(1) Distortion Control

f is expanding $\Rightarrow\ \exists\lambda > 1$, such that $\mathrm{D}f > \lambda$. f is $C^2 \Rightarrow \log \mathrm{D}f$ is C^1 and $\exists C > 0$, such that $\left|\log \mathrm{D}\tilde{f}\right| < C$. Thus, $\forall y_0, z_0 \in \mathbb{R}$, $y_i = \tilde{f}(y_0)$, $z_i = \tilde{f}(z_0)$, $i = 0, 1, \cdots, n$, we have:

$$\frac{\mathrm{D}\tilde{f}^n(y_0)}{\mathrm{D}\tilde{f}^n(z_0)} = \frac{\prod_{i=0}^{n-1} \mathrm{D}\tilde{f}(y_i)}{\prod_{i=0}^{n-1} \mathrm{D}\tilde{f}(z_i)}$$

$$\Rightarrow \qquad \left|\log \mathrm{D}\tilde{f}^n(y_0) - \log\ \mathrm{D}\tilde{f}^n(z_0)\right| \leq \sum_{i=0}^{n-1} \left|\log \mathrm{D}\tilde{f}(y_i) - \log \mathrm{D}\tilde{f}(z_i)\right|$$

$$\leq \quad C \cdot \sum_{i=0}^{n-1} d(y_i, z_i) \leq C \cdot \sum_{i=1}^{n} \lambda^{-i} \cdot d(y_n, z_n) \leq \tilde{C} \cdot d(y_n, z_n)$$

Here $\tilde{C} = \dfrac{C}{\lambda - 1}$. So we get:

Lemma 2.5. $\log \mathrm{D}\tilde{f}$ *is a* $\tilde{C}$*-Lipschitz function.* □

We will fix $y_n = y, z_n = z$ in the following text, and let x_0, y_0 and n be variables.

(2) Iteration of Lebesgue Measure

Let $\mu_n = f_*^n(\mathrm{Leb})$, $\nu_n = \dfrac{1}{n}\sum_{i=0}^{n-1}\mu_i$. Let $\varphi_n = \dfrac{\mathrm{d}\mu_n}{\mathrm{d\,Leb}}$, $\psi_n = \dfrac{\mathrm{d}\nu_n}{\mathrm{d\,Leb}} = \dfrac{1}{n}\sum_{i=0}^{n-1}\varphi_i$. We have the following lemma.

Lemma 2.6. $\log\varphi_n$ *and* $\log\psi_n$ *are* $\tilde{C}$*-Lipschitz functions.*

Proof.

$$\varphi_n(y) = \sum_{f^n(y')=y}\frac{1}{\mathrm{D}f^n(y')}.$$

Making use of the following elementary inequality,

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \sup_{1\leq i\leq n}\frac{a_i}{b_i}\ ,\ \forall a_i, b_i > 0,\ i = 1, 2, \cdots, n$$

we have:

$$\frac{\varphi_n(y)}{\varphi_n(z)} = \frac{\sum_{f^n(y')=y}\frac{1}{\mathrm{D}f^n(y')}}{\sum_{f^n(z')=z}\frac{1}{\mathrm{D}f^n(z')}} \leq \sup_{y',z'}\frac{\mathrm{D}f^n(z')}{\mathrm{D}f^n(y')}\ .$$

In the above and following expressions, y' and z', as subscipts under the sup symbol, are required to be in the same connected components of $f^{-n}([y,z])$. Then:

$$\left|\log\varphi_n(y) - \log\varphi_n(z)\right| \leq \sup_{y',z'}\left|\log\varphi_n(y') - \log\varphi_n(z')\right| \leq \tilde{C}d(y,z)$$

$$\left|\log\psi_n(y) - \log\psi_n(z)\right| = \left|\log\frac{\frac{1}{n}\sum_{i=0}^{n-1}\varphi_n(y)}{\frac{1}{n}\sum_{i=0}^{n-1}\varphi_n(z)}\right| \leq \left|\log\left(\sup_{0\leq i<n}\frac{\varphi_i(y)}{\varphi_i(z)}\right)\right| \leq \tilde{C}d(y,z)$$

So $\log\varphi_n$ and $\log\psi_n$ are $\tilde{C}$-Lipschitz functions. □

$\int_{S^1}\psi_n\,\mathrm{dLeb} = 1 \Rightarrow \exists y \in S^1,\ \psi_n(y) = 1 \Rightarrow \left|\log\psi_n\right| \leq \tilde{C}$, i.e. $\psi_n(z) \in [\mathrm{e}^{-\tilde{C}}, \mathrm{e}^{\tilde{C}}]$, $\forall z \in S^1$. Then by Arzela-Ascoli theorem, $\exists n_i \nearrow +\infty$ such that $\psi_{n_i} \to \psi$ uniformly as $i \to \infty$. Apparently $\log\psi$ is also a $\tilde{C}$-Lipschitz function. So $\lim_{n\to\infty}\nu_{n_i} = \nu$, where $\nu = \psi\,\mathrm{Leb}$ is equivalent to Leb.

(3) Ergodicity of ν

To prove ν is ergodic, only to show that:

$$\nu\left(S^1 \setminus \bigcup_{n=0}^{\infty} f^n(E)\right) = 0,\ \forall \nu(E) > 0.$$

Since Leb is equivalent to ν, we can take Leb instead of ν. For any $\mathrm{Leb}(E) > 0$, let $x \in E$ be a Lebesgue density point of E. Then $\forall \varepsilon > 0,\ \exists \delta_0 > 0$, such that:

$$\frac{\mathrm{Leb}([x-\delta, x+\delta] \setminus E)}{2\delta} < \epsilon,\ \forall 0 < \delta < \delta_0.$$

We can choose δ small enough and n accordingly, such that $f^n|_{[x-\delta,x+\delta]}$ is injective, and $\mathrm{Leb}\,(f^n([x-\delta, x+\delta])) > 1-\epsilon$. Then we have:

$$\begin{aligned}
\mathrm{Leb}\,(f^n([x-\delta, x+\delta])) &= \int_{[x-\delta,x+\delta]} \mathrm{D}f^n \,\mathrm{dLeb} \\
\mathrm{Leb}\,(f^n([x-\delta, x+\delta] \setminus E)) &= \int_{[x-\delta,x+\delta]\setminus E} \mathrm{D}f^n \,\mathrm{dLeb}
\end{aligned}$$

Since $\log \mathrm{D}f^n$ is $\tilde{C}$-Lipschitz, we have:

$$\frac{\mathrm{Leb}\,(f^n([x-\delta, x+\delta] \setminus E))}{\mathrm{Leb}\,(f^n([x-\delta, x+\delta]))} \le e^{2\tilde{C}} \frac{f^n([x-\delta, x+\delta] \setminus E)}{2\delta} < e^{2\tilde{C}}\epsilon.$$

Hence $\mathrm{Leb}(f^n(E)) > (1-\epsilon)(1-e^{2\tilde{C}}\epsilon)$. For ϵ is arbitrary, we complete the proof of theorem 2.3 .

Remark. As a direct application of theorem 2.3, we consider the Smale's solenoid. According to corollary 1.11, the invariant measure ν_f on the solenoid induced by ν is also ergodic. Moreover, we can prove that almost every conditional measure of ν on its corresponding connected component, which is homeomorphic to S^1, is absolutely continuous.

3 Non-Uniformly Expanding Maps on S^1

In this section, we consider $C^{1+\theta}$ map $f : S^1 \circlearrowleft$, $Df > 0$. For every $x \in S^1$, we can define the so called lower Lyapunov exponent $\lambda_-(x) = \liminf_{n\to+\infty} \frac{1}{n} \log \mathrm{D}f^n(x)$.

Definition 3.1. f is called non-uniformly expanding, if $\exists X \subset S^1$ with $\mathrm{Leb}(X) > 0$, such that $\lambda_-(x) > 0, \forall x \in X$.

The main result in this section is the following theorem.

Theorem 3.1. *If f satisfies the conditions above, then there is an unique ergodic a.c.i.p. on S^1.*

The proof will be composed of the following four subsections.

3.1 Regularity of the Inverse Map

Let $\tilde{f} : \mathbb{R} \circlearrowleft$ be the natural lift of f mentioned before, then $\tilde{f}$ is a $C^{1+\theta}$-diffeomorphism of $\mathbb{R}$, and $\tilde{f}(x+1) = \tilde{f}(x) + d$, where d is the topological degree of f. Let $g = \tilde{f}^{-1}$, which is also a $C^{1+\theta}$-diffeomorphism of $\mathbb{R}$.

Lemma 3.2. *$\exists C > 0$, such that for every $x, y \in \mathbb{R}$ with $|x-y| < 1$, we have:*

$$\left|\log \mathrm{D}\tilde{f}(x) - \log \mathrm{D}\tilde{f}(y)\right| \leq C \cdot d(x,y)^\theta$$

$$\left|\log \mathrm{D}g(x) - \log \mathrm{D}g(y)\right| \leq C \cdot d(x,y)^\theta$$

Proof.

$$\tilde{f} \in C^{1+\theta} \quad \text{and} \quad (\log x)' = \frac{1}{x}$$

$$\Rightarrow \qquad \left|\log \mathrm{D}\tilde{f}(x) - \log \mathrm{D}\tilde{f}(y)\right| \leq \sup_{z\in\mathbb{R}} \frac{1}{\mathrm{D}\tilde{f}(z)} \cdot \left|\mathrm{D}\tilde{f}(x) - \mathrm{D}\tilde{f}(y)\right| \leq C_1 \cdot d(x,y)^\theta$$

$$g \in C^1 \quad \text{and} \quad \mathrm{D}g(x) = \frac{1}{\mathrm{D}\tilde{f}(g(x))}$$

$$\Rightarrow \quad \left|\log \mathrm{D}g(x) - \log \mathrm{D}g(y)\right| \leq \left|\log \mathrm{D}\tilde{f}(g(x)) - \log \mathrm{D}\tilde{f}(g(y))\right| \leq C_1 \cdot d(g(x), g(y))^\theta \leq C_2 \cdot d(x,y)^\theta$$

We can take $C = \max\{C_1, C_2\}$. □

3.2 Distortion Control at Hyperbolic Times

Lemma 3.3. *Given $\sigma > 1$, $0 < \varepsilon \leq \frac{1}{4}\log\sigma$, $\exists \delta > 0$, such that: if $x \in \mathbb{R}$ and $t \in \mathbb{N}$ satisfy $\left|\mathrm{D}g^i(x)\right| \leq \sigma^{-i}$, $i = 1, 2, \cdots, t$, then:*

$$\left|\log \mathrm{D}g^t(y) - \log \mathrm{D}g^t(z)\right| < C\bar{\sigma}d(y,z)^\theta < \epsilon, \ \forall y, z \in [x-\delta, x+\delta], \ i = 1, 2, \cdots, t.$$

Proof. Let $\bar{\sigma} = \sum_{j=0}^{\infty} \sigma^{-\frac{j\theta}{2}}$. Take $\delta \in (0, \frac{1}{2})$, such that $C\bar{\sigma}(2\delta)^\theta < \varepsilon$. When $i = 0$, the inequality holds automatically. Assume for $j = 1, 2, \cdots, i-1$, we have proved that

$$\left|\log \mathrm{D}g^j(y) - \log \mathrm{D}g^j(z)\right| < \epsilon, \ \forall y, z \in [x-\delta, x+\delta].$$

Then for $j = 1, 2, \cdots, i-1$:

$$\begin{aligned}
& \mathrm{D}g^j(x) \leq \sigma^{-j} \\
\Rightarrow \quad & \mathrm{D}g^j(w) < \exp\left(-j\log\sigma + \epsilon\right) < \sigma^{-\frac{j}{2}}, \ \forall w \in [x-\delta, x+\delta] \\
\Rightarrow \quad & d\left(g^j(y), g^j(z)\right) \leq \sup_{w\in[y,z]} \mathrm{D}g^j(w) \cdot d(y,z) < \sigma^{-\frac{j}{2}} d(y,z) \\
\Rightarrow \quad & \left|\log \mathrm{D}g^i(y) - \log \mathrm{D}g^i(z)\right| \leq \sum_{j=0}^{i-1} \left|\log \mathrm{D}g\left(g^j(y)\right) - \log \mathrm{D}g\left(g^j(z)\right)\right| \\
& \leq C \cdot d\left(g^j(y), g^j(z)\right)^\theta < C \cdot \sum_{j=0}^{i-1} \sigma^{-\frac{j\theta}{2}} \cdot d(y,z)^\theta < C\bar{\sigma}(2\delta)^\theta < \epsilon.
\end{aligned}$$

□

Definition 3.2. $t \in \mathbb{N}$ is called a σ-hyperbolic time for $x \in S^1$, if: $\mathrm{D}f^{t-j}\left(f^j(x)\right) \geq \sigma^{t-j},\ j = 0, 1, \cdots, t-1$.

Corollary 3.4. *If t is a σ-hyperbolic time for x, and $\varphi_t = \dfrac{\mathrm{d}f^t_*(\mathrm{d\,Leb})}{\mathrm{d\,Leb}}$, then:*

$$\left|\log\varphi_t(y) - \log\varphi_t(z)\right| < C\bar{\sigma}d(y,z)^\theta < \epsilon,\ \forall y, z \in [\tilde{f}^t(x) - \delta, \tilde{f}^t(x) + \delta].$$

Proof. Noticing that $\varphi_n(y) = \sum\limits_{f^n(y')=y} \dfrac{1}{\mathrm{D}f^n(y')}$ and applying lemma 3.3, to get the conclusion we just need to do the same as in lemma 2.6. □

3.3 Existence of Hyperbolic Times

Definition 3.3. Given $A > 0,\ a_i \leq A, i \in \mathbb{N}$ and $c > 0$, we say i is a c-top time, if: $\sum\limits_{k=j+1}^{i} a_k \geq c(i-j),\ j = 0, 1 \cdots, i-1$.

Lemma 3.5 (Pliss)**.** *Given $c < b < A$ and $i_0 \in \mathbb{N}$, suppose that $\sum\limits_{k=1}^{i_0} a_k \geq b\, i_0$. Then:*

$$\#\left\{1 \leq i \leq i_0 \,\middle|\, i \text{ is a } c\text{-top time}\right\} \geq \frac{b-c}{A-c} \cdot i_0.$$

Proof. Let $\tilde{a}_k = a_k - c \leq A - c$, $\sum\limits_{k=1}^{i_0} \tilde{a}_k \geq (b-c)i_0$. i is a c-top time $\Leftrightarrow \sum\limits_{k=j+1}^{i} \tilde{a}_k \geq 0,\ j = 0, 1 \cdots, i-1$. Let $s_j = \sum\limits_{k=1}^{j} \tilde{a}_k$, so i is a c-top time $\Leftrightarrow s_i \geq s_j,\ j = 1, 2, \cdots, i-1$. Therefore if $j < i$ are two adjacent c-top times, then $s_j \geq s_{i-1}$ and hence $s_j + \tilde{a}_i \geq s_i$. So we get:

$$\#\left\{1 \leq i \leq i_0 \,\middle|\, i \text{ is a } c\text{-top time}\right\} \times (A-c) \geq \sum_{\substack{1 \leq i \leq i_0 \\ i \text{ is a} \\ c\text{-top time}}} \tilde{a}_i \geq \sum_{i=1}^{i_0} \tilde{a}_i \geq (b-c)i_0.$$

□

Since $\mathrm{Leb}(X) > 0$ and $\forall x \in X,\ \lambda_-(x) > 0$, then $\exists Y \subset X,\ N \in \mathbb{N}$ and $\lambda > 0$, such that $\beta = \mathrm{Leb}(Y) > 0$ and $\forall x \in Y, n > N, \dfrac{1}{n}\log \mathrm{D}f^n(x) > \lambda$. If we choose σ so small that $\log\sigma < b$, take $A = \max\limits_{x \in S^1} \log \mathrm{D}f(x)$, $b = \lambda$, $c = \log\sigma$, and denote $\alpha = \dfrac{b-c}{A-c}$, then we get:

Corollary 3.6. *$\forall x \in Y,\ n > N$, we have at least $\alpha \cdot n$ σ-hyperbolic times in* $\{1, 2, \cdots, n\}$. □

The parameters σ, λ, etc will be fixed in the following text.

3.4 Iteration of Lebesgue Measure

In the following text we will denote measures on $\mathbb{R}$ by symbols with superscript '$\sim$' and the corresponding ones on S^1 by the same symbols without '$\sim$'. For all $n \geq 0$, we denote $\tilde{\mu}_n = \tilde{f}_*^n \left(\text{Leb}\big|_{[0,1)}\right)$, $\mu_n = \pi_* \tilde{\mu}_n = f_*^n(\text{Leb})$ and $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} \mu_i$. From now on, we fix δ given in lemma 3.3 such that $\frac{1}{\delta} \in \mathbb{N}$ and fix $\varepsilon = \frac{1}{4}\log\sigma$. We denote by $\tilde{\eta}_i$ the restriction of $\tilde{\mu}_i$ on the union of intervals of the form $[k\delta, (k+1)\delta]$, $k \in \mathbb{Z}$ such that $\exists x \in [0,1) \cap \pi^{-1}(Y)$, $\tilde{f}^n(x) \in [k\delta, (k+1)\delta]$ and i is a σ-hyperbolic time for x. Let $\eta_i = \pi_* \tilde{\eta}_i$ and $\gamma_i = \frac{1}{i}\sum_{j=0}^{i-1} \eta_j$. Notice that $\tilde{\eta}_i$, $\eta_i$and γ_i are generally not probability measures.

It is clear that the support of $\eta_i$and γ_i are union of intervals of the form $[k\delta, (k+1)\delta]$, and $\eta_i, \gamma_i << \text{Leb}$. Suppose $\eta_i = \phi_i \,\text{Leb}$ and $\gamma_i = \psi_i \,\text{Leb}$, then the following lemma holds:

Lemma 3.7. $\log\phi_i$ *and* $\log\psi_i$ *are* $(C\bar{\sigma}, \theta)$*-Hölder continuous on each interval* $[k\delta, (k+1)\delta]$ *contained in the support of* η_i *and* γ_i *respectively. Furthermore,* $\forall x, y \in [k\delta, (k+1)\delta]$, *we have:*

$$\left|\log\phi_i(x) - \log\phi_i(y)\right| < \epsilon \quad \textit{and} \quad \left|\log\psi_i(x) - \log\psi_i(y)\right| < \epsilon$$

Proof. Let $\tilde{\eta}_i = \tilde{\phi}_i \,\text{Leb}$. Corollary 3.4 tells us that $\log\tilde{\phi}_i$ is $(C\bar{\sigma}, \theta)$-Hölder continuous on each interval $[k\delta, (k+1)\delta] \subset \text{supp}\,\tilde{\eta}_i$. Suppose $[k\delta, (k+1)\delta] \subset \text{supp}\,\eta_i$.

$$\pi^{-1}([k\delta, (k+1)\delta]) = \bigcup_{n\in\mathbb{Z}} [k\delta + n, (k+1)\delta + n] \Rightarrow \phi_i\big|_{[k\delta,(k+1)\delta]} = \sum_j \tilde{\phi}_i\big|_{[k\delta+n_j,(k+1)\delta+n_j]}.$$

Here n_j's are such that $[k\delta + n_j, (k+1)\delta + n_j] \subset \text{supp}\,\tilde{\eta}_i$. Following what we did in lemma 2.6, we can get the conclusion. □

Lemma 3.8. $\forall n \in \mathbb{N}$, $\gamma_n(S^1) > (1 - \frac{1}{n})\alpha\beta$.

Proof. By definition of γ_i and η_i, we have:

$$\begin{aligned}
\gamma_n(S^1) &= \frac{1}{n}\sum_{i=0}^{n-1} \eta_i(S^1) \geq \frac{1}{n}\sum_{i=0}^{n-1} \text{Leb}\left(\left\{ x \in Y \;\middle|\; i \text{ is a } \sigma\text{- hyperbolic time for } x \right\}\right) \\
&= \frac{1}{n}\int_Y \#\left\{ 0 \leq i < n \;\middle|\; i \text{ is a } \sigma\text{-hyperbolic time for } x \right\} \text{dLeb}(x) \\
&\geq \frac{1}{n}\int_Y \alpha(n-1)\,\text{dLeb} = (1 - \frac{1}{n})\alpha\beta\,.
\end{aligned}$$

The last inequality is due to corollary 3.6. □

By compactness, $\exists n_i \nearrow +\infty$, such that $\lim_{i\to\infty} \nu_{n_i} = \nu$ and $\lim_{i\to\infty} \gamma_{n_i} = \gamma$. It is clear that $f_*(\nu) = \nu$, $f_*(\gamma) = (\gamma)$ and $\gamma << \text{Leb}$, $\gamma(S^1) = \lim_{i\to\infty} \gamma_{n_i}(S^1) \geq \alpha\beta$. Moreover, $\gamma \leq \nu$, i.e. :

$$\int_{S^1} \varphi \,\text{d}\gamma \leq \int_{S^1} \varphi \,\text{d}\nu, \;\forall\varphi : S^1 \to [0, +\infty).$$

Consider the Lebesgue decomposition: $\nu = \nu^a + \nu^s$, where $\nu^a << \text{Leb}$ and $\nu^s \perp \text{Leb}$. $\gamma \leq \nu$, $\gamma << \text{Leb} \Rightarrow \gamma \leq \nu^a$, hence $\nu^a > 0$. On the other hand, $\nu = f_*\nu = f_*\nu^a + f_*\nu^s$. Notice that $\mathrm{D}f > 0 \Rightarrow f_*(\mathrm{Leb}) << \mathrm{Leb}$. Therefore, $f_*\nu^a << f_*(\mathrm{Leb}) << \mathrm{Leb}$. Next we show that $f_*\nu^s \perp \mathrm{Leb}$. This is because $\nu^s \perp \mathrm{Leb} \Rightarrow \exists E \subset S^1$, such that $\mathrm{Leb}(E) = \nu^s(E^c) = 0$, and therefore:

$$f_*(\nu^a)(E) = 0 \Rightarrow f_*\nu^s(E) = f_*\nu(E) = \nu(E) = \nu^s(E) = \nu^s(S^1) = f_*\nu^s(S^1) \Rightarrow f_*\nu^s(E^c) = 0.$$

So by the uniqueness of Lebesgue decomposition, we conclude that $f_*\nu^s = \nu^s$. Finally we obtain an a.c.i.p. $\nu_0 = \dfrac{1}{\nu^a(S^1)}\nu_a$,.

3.5 Erogodicity of the A.C.I.P.

Lemma 3.9. $\mathrm{Leb}\left(\bigcup_{n=0}^{\infty} f^n(Y)\right) = 1.$

Proof. Let $x \in Y$ be a Lebesgue density point of Y. Then

$$\lim_{\epsilon \to 0^+} \frac{\mathrm{Leb}\left([x-\epsilon, x+\epsilon] \setminus Y\right)}{2\epsilon} = 0.$$

Let $n_i \nearrow +\infty$ be a sequence of σ-hyperbolic times for x, and denote $f^{n_i}(x)$ by x_i. For each i, there exists $\delta_i > 0$, such that $f^{n_i}([x-\delta_i, x+\delta_i]) = [x_i - \delta, x_i + \delta]$. n_i's are σ-hyperbolic times $\Rightarrow \lim_{i\to\infty} \delta_i = 0$. We will follow what we did at the end of the proof of theorem 2.3 .
First, by lemma 3.3:

$$\frac{\mathrm{D}f^{n_i}(y)}{\mathrm{D}f^{n_i}(z)} \le \mathrm{e}^{2\epsilon}, \forall y, z \in [x-\delta_i, x+\delta_i].$$

Second:

$$\mathrm{Leb}\left([x_i-\delta, x_i+\delta] \setminus f^{n_i}(Y)\right) = \int_{[x-\delta_i, x+\delta_i]\setminus Y} \mathrm{D}f^{n_i}\,\mathrm{dLeb}$$

$$2\delta = \mathrm{Leb}\left([x_i-\delta, x_i+\delta]\right) = \int_{[x-\delta_i, x+\delta_i]} \mathrm{D}f^{n_i}\,\mathrm{dLeb}$$

Then we get:

$$\frac{\mathrm{Leb}\left([x_i-\delta, x_i+\delta] \setminus f^{n_i}(Y)\right)}{2\delta} \le e^{2\epsilon} \frac{\mathrm{Leb}\left([x-\delta_i, x+\delta_i] \setminus Y\right)}{2\delta_i} \to 0,\ i \to \infty.$$

Let x_∞ be some limit point of $\{x_i\}$. Without loss of generality we can assume that $\lim_{i\to\infty} x_i = x_\infty$. Denote $[x_\infty - \delta, x_\infty + \delta] \setminus \bigcup_{n=0}^{\infty} f^n(Y)$ by E. Then we have:

$$\mathrm{Leb}(E) \le \lim_{i\to\infty} \mathrm{Leb}\left([x_i-\delta, x_i+\delta] \setminus f^{n_i}(Y)\right) = 0$$

By the definition of Y, $\lambda_-(y) > 0$, a.e. $y \in [x_\infty - \delta, x_\infty + \delta]$. So in $\mathbb{R}$, the length of the interval $\tilde{f}^k([x_\infty - \delta, x_\infty + \delta]) \to \infty$ as $k \to \infty$, which means that $f^k([x_\infty - \delta, x_\infty + \delta]) = S^1$ for k sufficiently large. We choose such a k, finally:

$$\mathrm{D}f > 0\, \&\ \mathrm{Leb}(E) = 0 \Rightarrow \mathrm{Leb}(f^k(E)) = 0 \Rightarrow \mathrm{Leb}\left(\bigcup_{n=0}^{\infty} f^n(Y)\right) = 1$$

$\square$

Corollary 3.10. $\mathrm{Leb}\left(\left\{ x \in S^1 \,\middle|\, \lambda_-(x) > 0 \right\}\right) = 1$

Proof. The definition of Y tells us that $x \in \bigcup_{n=0}^{\infty} f^n(Y) \Rightarrow \lambda_-(x) > 0$. □

Corollary 3.11. $\forall X_1 \subset S^1,\ \mathrm{Leb}(X_1) > 0 \Rightarrow \mathrm{Leb}\left(\bigcup_{n=0}^{\infty} X_1\right) = 1.$

Proof. From corollary 3.10 we know that $\lambda_-(x) > 0,\ a.e.\ x \in X_1$. Then all the conclusions we have got about X still hold when it is replaced by X_1. So lemma 3.9 tells us that $\mathrm{Leb}\left(\bigcup_{n=0}^{\infty} f^n(X_1)\right) = 1.$ □

Noticing that $\nu_0 << \mathrm{Leb}$, we can conclude that ν_0 is ergodic from the corollary above at once. So we complete the proof of theorem 3.1 .

4 Kan's Example of SRB Measure

In this section we consider the dynamical system

$$f : S^1 \times [0,1] \circlearrowleft \ (r,s) \mapsto (3r, f_r(s)),$$

where f is of class $C^{1+\theta}$. If we regard $\pi : S^1 \times [0,1] \to S^1$ as a bundle with fiber $[0,1]$, where π is the bundle projection, then f is a fiber-preserving map. Moreover, we need several additional assumptions below:

- $0 < \mathrm{D}f_r < 3 \quad \forall r \in S^1$;
- $f_r(0) = 0,\ f_r(1) = 1 \quad \forall r \in S^1$;
- $\lambda_0 = \int_{S^1} \log \mathrm{D}f_r(0)\,\mathrm{d}r < 0,\ \lambda_1 = \int_{S^1} \log \mathrm{D}f_r(1)\,\mathrm{d}r < 0$;
- f_0 and $f_{\frac{1}{2}}$ has no fixed point on $(0,1)$;
- $f_0 - \mathrm{id} < 0,\ f_{\frac{1}{2}} - \mathrm{id} > 0 \quad$ on $(0,1)$;
- $\mathrm{D}f_0(0) < 1,\ \mathrm{D}f_{\frac{1}{2}}(1) < 1$.

From these assumptions we know that $(0,0)$ and $(\frac{1}{2}, 1)$ are hyperbolic fixed points of f. Let us denote by μ_i the Lebesgue measure on $S^1 \times \{i\}$, $i = 0,1$. Notice that μ_i is ergodic and we denote by $\mathcal{B}(\mu_i)$ the basin of μ_i. Recall that $\mathcal{B}(\mu_0) \cap \mathcal{B}(\mu_1) = \emptyset$.

The main result in this section is:

Theorem 4.1 (Kan)**.** *Under the given assumptions, we have:*

$$\mathrm{Leb}\left(S^1 \times [0,1] \setminus (\mathcal{B}(\mu_0) \cup \mathcal{B}(\mu_1))\right) = 0 \textit{ and } \overline{\mathcal{B}(\mu_i)} = S^1 \times [0,1],\ i = 0,1.$$

More precisely, we will actually prove that for every open set $U \subset S^1 \times [0,1]$, one has: $\mathrm{Leb}\,(U \cap \mathcal{B}(\mu_i)) > 0$, $i = 0,1$.

Definition 4.1 (Sinai-Ruelle-Bowen Measure)**.** Let $f : M \circlearrowleft$ be a smooth dynamical system on some compact Riemannian manifold M. $\mu \in P(M, f)$ is called an Sinai-Ruelle-Bowen measure, or simply SRB measure, if $\mathrm{Leb}\,(\mathcal{B}(\mu)) > 0$.

Remark. According to proposition 1.9 and definition, it is easy to see that an ergodic a.c.i.p. in $P(M, f)$ must be its unique SRB measure. So the a.c.i.p.'s we got in the sections before are examples of SRB measure.

So μ_0 and μ_1 are the unique SRB measures for f, and they are "intermingled".

We will take all the three subsections to prove this theorem. Due to the similarity between μ_0 and μ_1, we will only state and prove the results on ν_0 and omit the corresponding ones on μ_1 in subsections 4.1 and 4.2 .

4.1 Stable Manifold and SRB Measure

For every $x = (r, s) \in S^1 \times [0, 1]$, we consider the so called stable manifold

$$W^s(x) = \left\{ y \in S^1 \times [0, 1] \,\middle|\, \lim_{n\to+\infty} d(f^n(y), f^n(x)) = 0 \right\}$$

and local stable manifold

$$W^s_0(x) = \left\{ y \in \{r\} \times [0, 1] \,\middle|\, y \in W^s(x) \right\}.$$

$W^s_0(x)$ is an subinterval of $\{r\} \times [0, 1]$, since $f(\{r\} \times [s_1, s_2]) \subset \{3r\} \times [f_r(s_1), f_r(s_2)]$. We will always denote by l the length of subinterval of a fiber. Because

$$\lim_{n\to+\infty} d(f^n(y), f^n(x)) = 0 \iff \pi(f^n(y)) = \pi(f^n(x)), \text{ for some } n \in \mathbb{N},$$

we have $W^s(x) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(f^n\left(W^s_0(x)\right)\right)$. It is clear that for each $x \in S^1 \times \{0\}$ regular for μ_0, $W^s(x) \subset \mathcal{B}(\mu_0)$.

Proposition 4.2. *$\exists \delta > 0$ and $X_\delta \subset S^1 \times \{0\}$ with $\mu_0(X_\delta) > 0$, such that $l\left(W^s_0(x)\right) \geq \delta$.*

Proof. Let us denote by $\mathrm{D}^c f^n$ the nth derivative of f in the "central" direction, i.e. $\mathrm{D}^c f^n = \dfrac{\partial f^n}{\partial s}$. Then:

$$f^n(r, s) = (3^n r, f_{3^{n-1}r} \circ \cdots f_{3r} \circ f_r(s)) \Rightarrow \mathrm{D}^c f^n(r, s) = \prod_{i=0}^{n-1} \mathrm{D} f_{3^i r}(s_i).$$

Here $s_0 = s$, $s_{i+1} = f_{3^i r}(s_i)$, $i = 0, 1, \cdots, n-1$.

Lemma 4.3. *Given $\sigma > 1$, $\exists \delta > 0$ such that: if $x = (r, s)$ satisfies $\mathrm{D}^c f^n(x) < \sigma^{-n}$, $\forall n > 0$, then $\{r\} \times [0, \delta] \subset W^s_0(x)$.*

Proof. It is more or less a copy of lemma 3.3, with the same parameters $\epsilon, \delta, \sigma, C$ etc. So we need not to show all the details. First notice that $\log \mathrm{D} f_r$ is a (C, θ)-Hölder continuous function as we showed in lemma 3.2. Followed by the proof in lemma 3.3, we know that $\forall y \in \{r\} \times [0, \delta]$, $\left|\log \mathrm{D}^c f^n(y) - \log \mathrm{D}^c f^n(x)\right| < \dfrac{1}{4} \log \sigma$. Immediately we get that $l\,(f^n(\{r\} \times [0, \delta])) < \sigma^{\frac{-n}{2}} \delta$, which implies the conclusion. □

For every $x = (r, s) \in S^1 \times [0, 1]$, we introduce the upper central Lyapunov exponent:

$$\lambda_+^c(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{D}^c f^n(x).$$

Lemma 4.4. *Suppose $x \in S^1 \times \{0\}$ and $\sigma > 1$ are such that $\lambda_+^c(x) < -log\sigma < 0$. Then there is $i \in \mathbb{N}$ such that $\mathrm{D}^c f^n(f^i(x)) < \sigma^{-n}$, $\forall n \in \mathbb{N}$.*

Proof. Since $\lambda_+^c(x) < -log\sigma$, we know that $i = \sup\left\{ j \in \mathbb{N} \,\middle|\, \mathrm{D}^c f^j(x) \geq \sigma^{-j} \right\}$ is finite. The lemma holds for this i, because:

$$\mathrm{D}^c f^i(x) \cdot \mathrm{D}^c f^n\left(f^i(x)\right) \leq \sigma^{-(n+i)} = \mathrm{D}^c f^{n+i}(x) \Rightarrow \mathrm{D}^c f^n(f^i(x)) < \sigma^{-n}\ ,\ \forall n \in \mathbb{N}.$$

□

According to Birkhoff's ergodic theorem:

$$\lambda_+^c(x) = \int_{S^1} \log \mathrm{D} f_r(0)\, \mathrm{d}r = \lambda_0 < 0,\ \mu_0\text{-a.e. } x \in S^1 \times \{0\}.$$

Then fixing $\sigma = \mathrm{e}^{-\frac{\lambda_0}{2}}$, lemma 4.3 and 4.4 imply that for almost every $x \in S^1 \times \{0\}$, there is $i \in \mathbb{N}$ such that $l\left(W_0^s(f^i(x))\right) \geq \delta$. That is to say, if we define

$$X_\delta = \left\{ x \in S^1 \times \{0\} \,\middle|\, l\left(W_0^s(f^i(x))\right) \geq \delta \right\},$$

then $\mu_0\left(\bigcup_{i=0}^{\infty} f^{-i}(X_\delta)\right) = 1$. The ergodicity of μ_0 implies $\mu_0(X_\delta) > 0$, so we complete the proof of proposition 4.2. □

Since $\lambda_+^c(x) > 0$ for μ_0-a.e. $x \in S^1 \times \{0\}$, in fact in the the proof of proposition 4.2 we have proved that:

Corollary 4.5. $l\left(W_0^s(x)\right) > 0$, *$\mu_0$-a.e. $x \in S^1 \times \{0\}$.* □

According to the above corollary, Fubini's Theorem tells us that $\mathrm{Leb}\,(\mathcal{B}(\mu_0)) > 0$, i.e. μ_0 is a SRB measure.

4.2 Denseness of the Basins

To obtain that $\mathrm{Leb}\,(U \cap \mathcal{B}(\mu_0)) > 0$, for any open set $U \subset S^1 \times [0, 1]$, we only need to prove:

Proposition 4.6. $\mathrm{Leb}_{S^1\times\{s\}}\,([r-\epsilon, r+\epsilon] \times \{s\} \cap \mathcal{B}(\mu_0)) > 0,\ \forall \epsilon > 0, (r, s) \in S^1 \times (0, 1)$.

Proof. Due to corollary 4.5 and $\mathrm{D}^c f^n > 0$, we only need to prove the following lemma. □

Lemma 4.7. $\exists n \in \mathbb{N}$, *such that $f^n\,([r-\epsilon, r+\epsilon] \times \{s\})$ meets each interval $\{t\} \times [0, \delta]$.*

Proof. We fix some $n \in \mathbb{N}$ such that $3^n\epsilon > 1$, then the length of the projection to S^1 of the curve $f^n([r-\epsilon, r+\epsilon] \times \{s\})$ is large than 2. This implies that the curve intersect transversally with $W_0^s((0, 0)) = \{0\} \times [0, 1)$. Since $(0, 0)$ is a hyperbolic fixed point, the inclination lemma says that there is a subsequence of $\{\, f^n([r-\epsilon, r+\epsilon] \times \{s\})\,\}_{n\in\mathbb{N}}$ converging to $S^1 \times \{0\}$ in the sense of C^1-topology, which leads to the conclusion. □

4.3 Full Measure of the Basins

We remains to prove that $\mathrm{Leb}(Y) = 0$, where $Y = S^1 \times [0,1] \setminus (\mathcal{B}(\mu_0) \cup \mathcal{B}(\mu_1))$. In fact we can prove a stronger proposition which implies it:

Proposition 4.8. $\forall s \in (0,1),\ \mathrm{Leb}_{S^1\times\{s\}} Y = 0$.

First we need the following lemma and corollary. Notice that there is a condition $\mathrm{D}f_r < 3$ unused till now.

Lemma 4.9. *$\exists \tau > 0$ such that: $\forall x = (r,s) \in S^1 \times [0,1]$, $\forall v \in T_x\left(S^1 \times [0,1]\right)$, $v = v^r \frac{\partial}{\partial r} + v^s \frac{\partial}{\partial s}$ and $w = \mathrm{D}f(v) = w^r \frac{\partial}{\partial r} + w^s \frac{\partial}{\partial s}$, we have $|v^s| \le \tau |v^r| \Rightarrow |w^s| \le \tau |w^r|$.*

Proof. It only needs some calculation. First we have:

$$\mathrm{D}f(v) = v^r \mathrm{D}f\left(\frac{\partial}{\partial r}\right) + v^s \mathrm{D}f\left(\frac{\partial}{\partial s}\right) = 3v^r \frac{\partial}{\partial r} + \left(v^r \frac{\partial f_r}{\partial r} + v^s \mathrm{D}^c f\right)\frac{\partial}{\partial s} = w^r \frac{\partial}{\partial r} + w^s \frac{\partial}{\partial s}.$$

Let α, β be universal positive constants such that $0 < \mathrm{D}^c f < \alpha$ and $\left|\frac{\partial f_r}{\partial r}\right| < \beta$, and in particular we can choose $\alpha < 3$. Taking $\tau = \frac{\beta}{3-\alpha} > 0$, then we have:

$$|v^s| \le \tau |v^r| \Rightarrow |w^s| = |v^r \frac{\partial f_r}{\partial r} + v^s \mathrm{D}^c f| \le \beta |v^r| + \alpha |v^s| \le (\tau\alpha + \beta)|v^r| = \tau |w^r|.$$

□

This lemma implies the following corollary immediately.

Corollary 4.10. *If the graph of a curve $\gamma : [0,1] \to S^1 \times [0,1]$ is τ-Lipschitz when it is regarded as a function from some segment in S^1 to $[0,1]$, so is $f(\gamma)$.* □

Proof of proposition 4.8. We use reduction to absurdity and suppose that $\exists s \in (0,1)$, such that $\mathrm{Leb}\left(S^1 \times \{s\} \cap Y\right) > 0$. Then $\exists r \in S^1$, such that (r,s) is a Lebesgue density point of $S^1 \times \{s\} \cap Y$. We denote $\gamma_n = [r - \frac{1}{3^n}, r + \frac{1}{3^n}] \times \{s\}$ endowed with probability $\eta_n = \frac{3^n}{2} \cdot \mathrm{Leb}_{\gamma_n}$. We will consider its nth iteration $\Gamma_n = f^n(\gamma_n)$, the corresponding probability $\nu_n = f^n_*(\eta_n)$, and the length of whose projection curve to S^1 is 2.

Firstly, noticing that $f^{-n}(Y) = Y$, then the Lebesgue density point theorem implies that $\lim_{n\to\infty} \eta_n (\gamma_n \cap Y) = 1$. As we did in the last section, the uniform distortion control for all $f^n\big|_{\gamma_n}$ tells us that, roughly speaking, ν_n on Γ_n are uniformly "equivalent" to η_n, and therefore $\lim_{n\to\infty} \nu_n (\Gamma_n \cap Y) = 1$. Here the details are omitted to save trouble.

Secondly, by corollary 4.10, Γ_n is the graph of some τ-Lipschitz function φ_n. So by Aezela-Ascoli theorem, there is an subsequence of $\{\varphi_n\}_{n\in\mathbb{N}}$ converging to a τ-Lipschitz function φ uniformly, which is the graph of some curve Γ with projective length 2. If $\Gamma \subset S^1 \times \{0\}$, then $\exists n$ large enough, such that $\Gamma_n \subset S^1 \times [0,\delta]$, where δ is given in proposition 4.2. So due to proposition 4.2, since φ_n is τ-Lipschitz, we can get that $\nu_n (\Gamma_n \cap \mathcal{B}(\mu_0)) > \mu_0 (X_\delta)$, which makes a contradiction. If $\Gamma \subset S^1 \times \{1\}$, the argument is all the same. Otherwise, $\Gamma \not\subseteq S^1 \times \{0,1\}$. In this case Γ intersects transversally with $W^s_0((0,0))$, so as in lemma 4.7, the inclination lemma tells us that there exist a subsegment $\tilde{\Gamma}$ of Γ and $m \in \mathbb{N}$, such that

$f^m(\tilde{\Gamma}) \subset S^1 \times [0, \dfrac{\delta}{2}]$, and the length of whose projective curve is 1. Instead of Γ, an argument on $f^m(\tilde{\Gamma})$ similar to the above one leads to a contradiction again, so we complete the proof. □

Up to now we have finished the whole proof of theorem 4.1 .

Exercise 4.1. *In theorem 4.1, to obtain the same conclusion, the smoothness of f can be reduced to C^1.*

Clue to the proof. Review the whole proof of theorem 4.1, we only need to prove that lemma 4.3 still holds when f is only C^1. This can be achieved if we make use of absolute continuity of $\log \mathrm{D}^c f$ instead of Hölder continuity. □

Exercise 4.2. *$f : S^1 \circlearrowleft$ is of class $C^{1+\theta}$ or even C^1 with $\mathrm{D}f > 0$, and μ is an ergodic invariant measure for f such that $\int_{S^1} \log \mathrm{D}f \, \mathrm{d}\mu < 0$. Then μ is a Dirac measure supported on some sink, i.e. an attracting periodic orbit.*

5 SRB Measures for Partially Hyperbolic Systems Whose Central Direction Is Mostly Contracting

5.1 Partial Hyperbolicity

Let (M, g) be a compact Riemannian manifold and $f : M \circlearrowleft$ is a diffeomorphism. $\|\cdot\|$ denotes the norm on M induced by g. A splitting of the tangent bundle, denoted by $\mathrm{T}M = E \oplus F$, is called $\mathrm{D}f$-invariant, or invariant shortly, if $\mathrm{D}f(E_x) = E_{f(x)}$ and $\mathrm{D}f(F_x) = F_{f(x)},\ \forall x \in M$.

Let us denote by $\mathrm{D}f\big|_{E_x}$ the restriction of $\mathrm{D}f$ to E_x for each $x \in M$, and it is similar to$F$.

Definition 5.1. An compact invariant subset K of M is called partially hyperbolic, if the tangent bundle restricted to K has a $\mathrm{D}f$-invariant splitting $T_K M = E \oplus F$, such that $\exists C > 1$ and $0 < \lambda < 1$, for each $x \in K$ and $n \in \mathbb{N}$, we have:

- $\left\|\mathrm{D}f^n\big|_{E_x}\right\| \cdot \left\|\mathrm{D}f^{-n}\big|_{F_{f^n(x)}}\right\| \le C\lambda^n$, i.e. E is dominated by F;
- either $\left\|\mathrm{D}f^n\big|_{E_x}\right\| \le C\lambda^n$, when E is uniformly contracting,

  or $\left\|\mathrm{D}f^{-n}\big|_{F_x}\right\| \le C\lambda^n$, when F is uniformly expanding.

In the first case we denote $E = E^s$ and $F = E^{cu}$; in the second one $E = E^{cs}$ and $F = E^u$. Here s = stable, c = center, and u = unstable.

Remark. In the above definition, the splitting of $\mathrm{T}M$ is usually not unique. For instance, when $\mathrm{T}_K M = E^s \oplus E^c \oplus E^u$.

Proposition 5.1. *In the above definition we can always choose another norm $|\cdot|$ on M equivalent to $\|\cdot\|$, such that $C = 1$. Such a norm is called adapted to the partially hyperbolic structure.*

Proof. Taking any $\tau \in (\lambda, 1)$ and $N \in \mathbb{N}$ such that $N \geq \dfrac{\log \tau - \log \lambda}{\log C}$, we define:

$$|v| = \sum_{i=0}^{N-1} \tau^{-i} \left\| \mathrm{D}f^{i}(v) \right\|, \ v \in E \quad \text{and} \quad |v| = \sum_{i=0}^{N-1} \tau^{i} \left\| \mathrm{D}f^{-i}(v) \right\|, \ v \in F\,.$$

It is easy to verify that $|\mathrm{D}f(v)| \leq \tau|v|,\ v \in E$ and $|\mathrm{D}f^{-1}(v)| \leq \tau|v|,\ v \in F$. So we can define:

$$|v|^2 = |v_{_E}|^2 + |v_{_F}|^2,\ \forall v = v_{_E} + v_{_F} \in E \oplus F\,.$$

Clearly this new norm is equivalent to the original one. □

According to the proposition above, we will always assume the norm $\|\cdot\|$ is adapted.

Proposition 5.2. *The partially hyperbolic structure on K is robust under perturbation, i.e. there exists an open neighborhood U of K, such that for any C^1-diffeomorphism $g : U \to g(U) \subset M$ sufficiently close to f in C^1-topology, if $\tilde{K} \subset U$ is an compact invariant set for g, then $\tilde{K}$ is also partially hyperbolic.*

Clue to proof. Fixing $\alpha > 0$, for each $x \in K$, we introduce a cone at x:

$$C_{\alpha}^{F}(x) = \left\{ v = v_{_E} + v_{_F} \in E \oplus F \,\middle|\, \|v_{_F}\| \geq \alpha \|v_{_E}\| \right\}.$$

Clearly it is forward invariant, i.e. $\mathrm{D}f\left(C_{\alpha}^{F}(x)\right) \subset C_{\alpha}^{F}(f(x))$. It is easy to see that:

$$E^{u}(x) = \bigcap_{n=0}^{\infty} \mathrm{D}f^{n}_{f^{-n}(x)}\left(C_{\alpha}^{F}(f^{-n}(x))\right)\,.$$

We can prove that if g is a small C^1-perturbation of f, then the right hand of the above equality also represents the unstable subbundle for g, when f is replaced by g in the expression. The analogous discussion on E^{cs} is just the same. □

5.2 Invariant Foliations

From now on we assume that f is of class C^2 and there is a partially hyperbolic splitting $\mathrm{T}_K M = E^{cs} \oplus E^{u}$ over K. Moreover, K is an topological attractor, i.e. there is an open neighborhood U of K such that $f(\overline{U}) \subset U$ and $K = \bigcap_{n=0}^{\infty} f^{n}(U)$. Then the following theorem holds:

Theorem 5.3 (Brin & Pesin)**.** *K admits a foliation structure whose leaves are C^2-embedded submanifolds tangent to E^u, and the curvatures of the leaves are uniformly bounded.*

Remark. Locally K is of the form $C \times D^{u}$, where $C \subset \mathrm{D}^{cs}$ is compact, D^{cs} is a disk tangent to E^{cs} and D^{u} is similar.

5.3 Distortion Control in the Unstable Direction

We introduce a new function $J^u f^n$, which is defined as $J^u f^n(x) = \left|\det Df^n\big|_{E^u(x)}\right|$.

Lemma 5.4. *There is $\tau > 0$ such that for each $n \in \mathbb{N}$, $\log J^u f^{-n}$ is τ-Lipschitz as a function on any certain unstable leaf L of the foliation of K.*

Proof. Let us denote by d^u the distance on each leaf. Because E^u is uniformly expanding, $d^u(f^{-n}(x), f^{-n}(y)) < \lambda^n d^u(x, y)$, $\forall x, y \in L$ close enough. Compared with lemma 2.5, $J^u f^{-n}$ here is the analogue of $\tilde{f}$ there. So we can follow the proof before. □

Corollary 5.5. *As functions on $L \times L$, we have:*

$$\lim_{n\to\infty} (\log J^u f^{-n}(x) - \log J^u f^{-n}(y)) = \log \psi(x, y),\ \forall x, y \in L\,.$$

Moreover, $\log \psi$ is also τ-Lipschitz. □

5.4 Measures Absolutely Continuous in the Unstable Direction

Let μ be a probability on M with $\operatorname{supp}\mu \subset K$, and $C \times D^u$ is a local chart of K. Then there is a positive measure η on C, which is the projection of $\mu\big|_{C\times D^u}$ to C, and a measure decomposition:

$$\mu\big|_{C\times D^u} = \int_C \mu_\xi \, d\eta(\xi)\,.$$

Here ξ denotes the local unstable leaf and μ_ξ is the corresponding conditional measure with respect to μ.

Definition 5.2. Under the above conditions, μ is called absolutely continuous in the unstable direction, if for η-a.e. ξ, $\mu_\xi << \mathrm{Leb}_\xi$.

Definition 5.3 (u-Gibbs State)**.** Let μ be absolutely continuous in the unstable direction and $C \times D^u$ be a local chart, we say μ is a u-Gibbs state, if η-a.e. $\xi \in C$, $\mu_\xi = \varphi_\xi \,\mathrm{Leb}_\xi$ and $\dfrac{\varphi_\xi(x)}{\varphi_\xi(y)} = \psi(x, y)$, $\forall x, y \in \xi$.

Lemma 5.6. *Every invariant probability absolutely continuous in the unstable direction is a u-Gibbs state.*

Lemma 5.7. *For any $D \subset U$ being a C^2-embedded disk transversal to E^{cs} and of dimension* $\dim E^u$*, let μ_0 be the normalized Lebesgue measure on D, and $\mu_n = f_*\mu_0$, $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} \mu_i$. Then every limit point ν of $\{\nu_n\}_{n\in\mathbb{N}}$ is an invariant probability absolutely continuous in the unstable direction, and hence a u-Gibbs state.*

The main difficulty here is the following lemma.

Lemma 5.8. *The curvatures of the disks $f^n(D)$ are uniformly bounded.*

Lemma 5.9. *Let ν be a u-Gibbs state. Then in the sense of ergodic decomposition theorem, almost all its ergodic components are u-Gibbs states.*

Proof. Let $R_\nu = \left\{x \in K \,\middle|\, x \text{ is regular for } \nu\right\}$. Then $\nu(R_\nu) = 1$, and $\mathrm{Leb}_{L_x}(L_x \setminus R_\nu) = 0$, for ν-a.e. $x \in K$, since ν is a u-Gibbs state. Here L_x denotes the unstable leaf containing x, and such an L_x is regular, i.e. :

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(y)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}\ ,\ \forall y \in R_\nu \cap L_x.$$

Since the above limit is an ergodic measure, the support of almost every ergodic component of ν is a union of entire leaves, which implies the conclusion. □

In fact we have the following much stronger result.

Theorem 5.10 (Bonatti & Viana)**.** *For* Leb*-a.e. $x \in U$, any limit point of* $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \right\}_{n\in\mathbb{N}}$ *is a u-Gibbs state.*

5.5 Systems Whose Central-Stable Direction Is Mostly Contracting

By mostly contracting, we mean that there is a nonuniform contraction in the central-stable direction.

Theorem 5.11 (Bonatti & Viana)**.** *Assume that for every unstable leaf L there is a subset X with* $\mathrm{Leb}_L(X) > 0$*, such that $\forall x \in X$, its upper central-stable Lyapunov exponent*

$$\lambda_+^{cs}(x) = \limsup_{n\to+\infty} \frac{1}{n} \log \left\| \mathrm{D}f^n\big|_{E^{cs}(x)} \right\| < 0\,.$$

Then we have:

(1) there are only finitely many ergodic u-Gibbs states $\mu_1, \cdots, \mu_n$;

(2) each μ_i is an SRB measure;

(3) $\mathrm{Leb}\left(K \setminus \bigcup_{i=1}^{n} \mathcal{B}(\mu_i)\right) = 0.$

Clue to proof. $\forall x \in L$ with $\lambda_+^{cs}(x) < 0$, there is a C^2-embedded disk $W_0^{cs}(x)$, the so called "Pesin stable manifold", tangent to E^{cs} and $y \in W_0^{cs}(x) \Rightarrow \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0$. As a result, for each unstable leaf L, there are $\delta > 0$ and $X \subset L$ with $\mathrm{Leb}_L(X) > 0$, such that $\forall x \in X$, the radius of $W_0^{cs}(x)$ is large than δ. According to a theorem of Pugh & Shub, when f is C^2, the foliation of local central-stable manifolds is absolutely continuous.

Therefore, if μ is an ergodic u-Gibbs state, then we can take a regular leaf L for μ and a corresponding $X \subset L$ as above. So we have:

$$\mathcal{B}(\mu) \supset \bigcup_{x\in X\cap\mathcal{B}(\mu)} W_0^{cs}(x),$$

and the latter is of positive Lebesgue measure, i.e. μ is an SRB measure.

If there are infinitely many ergodic u-Gibbs states, then we can pick a convergent sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of them, such that $\lim_{n\to\infty} \mu_n = \mu$. Then μ is also a u-Gibbs state, and so is one of its ergodic components ν. Let $L \subset \operatorname{supp} \nu \subset \operatorname{supp} \mu$ be an unstable leaf. Then when n is large enough, $L \subset \operatorname{supp} \mu_n$. It contradicts to the assumption that μ_n's are distinct, since they are all ergodic.

To prove $\mathrm{Leb}\left(K \setminus \bigcup_{i=1}^{n} \mathcal{B}(\mu_i)\right) = 0$, we can follow what we did in subsection 4.3, here we do not show the details again. $\square$